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Irreversible Investment and Knichtian Uncertainty

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IRREVERSIBLE INVESTMENT AND KNIGHTIAN UNCERTAINTY*

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Abstract

When firms decide about irreversible investment, they may not have perfect confidence about their perceived probability measure describing future uncertainty. They may think other probability measures perturbed from the original one are also probable. Uncertainty characterized by not a single probability measure but a set of probability measures is called Knightian uncertainty. The effect of Knightian uncertainty on the value of irreversible investment opportunity is shown to be drastically different from that of the traditional uncertainty in the form of risk. Specifically, an increase in Knightian uncertainty decreases the value of investment opportunity while an increase in risk increases it.

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1. Introduction and Summary

Investment decisions of firms typically involve three features. First, future market conditions are uncertain. Second, costs of investment are sunk and thus investment is irreversible. Third, investment opportunity does not vanish at once and when to invest becomes a critical decision. This irreversibility of investment under uncertainty and resulting optimal investment timing problem have attracted considerable attention in recent years, especially after McDonald and Siegel (1986) successfully applied financial option pricing techniques to this problem and Dixit and Pindyck (1994) related option-theoretic results to neoclassical investment theory.

Most irreversible investment studies, however, assume more than that future market conditions are uncertain. In these studies future uncertainty is characterized by a certain probability measure over states of nature. This amounts to assume that the firm is *perfectly certain* that future market conditions are governed by this particular probability measure. However, this assumption may be too farfetched: the firm may not be so sure about future uncertainty. It may think other probability measures are also likely and have no idea of relative “plausibility” of these measures. Uncertainty that is *not* reducible to a single probability measure and thus characterized by a *set* of probability measures is often called *Knightian uncertainty* (see Knight, 1921, and also see Keynes, 1921, 1936), or ambiguity in some cases. In contrast, the uncertainty which *is* reducible to a single probability measure with known parameters is referred to as *risk*. That is, the firm may face Knightian uncertainty in contemplating its investment, facing not a single probability measure but a set of probability measures.

The purpose of this paper is to show the effect of uncertainty on the value of irreversible investment opportunity drastically different between risk and Knightian uncertainty. Specifically, the standard result that an increase in uncertainty increases the value of an irreversible investment opportunity, is reversed if uncertainty is not risk but Knightian uncertainty. That is, an increase in Knightian uncertainty (properly defined) reduces the value of an irreversible investment opportunity, while the opposite is true for an increase in risk in the form of an increase in variance. In contrast, both of them have the same effect on the value of waiting: they increase the value of waiting and make waiting more likely.

In this paper, we take a patent as an example of irreversible investment. To highlight the effect of Knightian uncertainty, the firm is assumed to be risk-neutral but *uncertainty-averse* in the sense that it computes the expected profit by using the "worst" element in the set of the probability measures characterizing Knightian uncertainty and chooses its strategy to maximize it (maximin criterion).[1]

Following the standard procedure of irreversible investment studies, we assume that (1) to utilize a patent, the firm has to build a factory and this construction cost is sunk after its completion, and (2) the profit flow after the construction is characterized by a geometric Brownian motion with a drift. Then, the firm first calculates the value of the utilized patent, and then contemplates when to build a factory taking account of the value of the utilized patent and the cost of investment. The firm's problem is thus formulated as an optimal stopping problem in continuous time.[2]

Unlike in the standard case, however, we assume that the firm is not perfectly certain that the profit flow is generated by a particular geometric Brownian motion with say, variance σ^2 and drift μ, or equivalently, by a probability measure underlying this geometric Brownian motion, say P. The firm may think that the profit flow is generated by other probability measures slightly different from P. The firm has no idea about which of these probability measures is "true." Thus, the firm faces Knightian uncertainty with respect to probability measures characterizing the profit flow.

We assume that the firm thinks these probability measures are not far from P. Firstly, we assume that these probability measures agree with P with respect to zero probability events. (That is, if a particular event's probability is zero with P, then it is also zero with these probability measures.) Then, these probability measures can be shown as a perturbation of P by a particular "density generator." Second, the deviation of these probabilities from P is not large in the sense that the corresponding density generator's move is confined in a range $[-\kappa, \kappa]$, where κ can be described as a degree of this Knightian uncertainty. This specification

[1]For axiomatization of such a behavior, see Schmeidler (1989), Gilboa (1987) and Gilboa and Schmeidler (1989).

[2]The standard procedure is to apply financial option pricing techniques to this problem, exploiting the fact that an *un*-utilized patent can be considered as a call option whose primal asset is a *utilized* patent which generates a stochastic flow of profits, and whose exercise price is a fixed cost of building a factory to produce patented products. (For example, see Dixit and Pindyck, 1994.) This approach and the optimal stopping approach are two ways of formulating the same problem and produce the same result.

of Knightian uncertainty in continuous time is called κ-ignorance in Chen and Epstein (2001) in a different context.

These two assumptions, though they seem quite general, have strong implications. Under the first assumption, for each of probability measures constituting the firm's Knightian uncertainty, the profit flow is characterized by a "geometric Brownian motion" of the same variance σ^2 with respect to this probability measure. Thus, "geometric Brownian motions" corresponding to these probability measures are different only in the drift term. (In fact, this is a direct consequence of well-known Girzanov's Theorem in the literature of mathematical finance. See for example, Karatzas and Shreve, 1991). Under the second assumption, the minimum drift term becomes $\mu - \kappa\sigma$ among them. Note that the uncertainty-averse firm evaluates the present value of the patent according to the "worst" scenario. Loosely speaking, this amounts to calculating the patent's value using the probability measure corresponding to the geometric Brownian motion with variance σ^2 and the minimum drift $\mu - \kappa\sigma$. Thus, an increase in κ, the degree of Knightian uncertainty, leads to a lower value of the utilized patent at the time of investment, since it is evaluated by less favorable Brownian motion process governing the profit flow from the utilized patent. Consequently, the value of the unutilized patent is also reduced.

This is in sharp contrast with a positive effect of an increase in risk (that is, an increase in σ) on the value of the unutilized patent, when there is no Knightian uncertainty. An increase in σ under no Knightian uncertainty implies, when the firm waits, it can undertake investment only when market conditions are more favorable than before (since it does not have to undertake investment when market conditions are less favorable). Consequently, an increase in σ increases the value of the unutilized patent.

In contrast, both an increase in risk and that in Knightian uncertainty raise the value of waiting and thus make the firm more likely to postpone investment. However, the reason for waiting is different between the two. As explained earlier, an increase in risk (σ) under no Knightian uncertainty leaves the value of utilized patent unchanged but increases the value of unutilized patent, and thus makes waiting more profitable. An increase in Knightian uncertainty (κ) reduces both the value of the utilized patent and that of the unutilized patent but it lowers the former more than the latter. This is because the value of the unutilized patent depends not only on proceeds from undertaking investment (the utilized patent) but also on proceeds from not undertaking investment, which is independent of the value of the utilized patent. Since the value of the utilized patent is reduced more than that of the unutilized patent, the firm finds waiting more profitable. Intuitively speaking, an increase in Knightian uncertainty makes an uncertainty-averse decision-maker more likely to postpone uncertain investment.

This paper is organized as follows. In Section 2, we present a simple two-period two-state example and explain intuitions behind the result of this paper. In Section 3, we formulate the firm's irreversible investment problem in continuous time. In the same section, we formally define Knightian uncertainty in continuous time, derive an explicit formula of a utilized patent, and investigate the optimal investment timing problem. In Section 4, we conduct a sensitivity analysis and present the main result of this paper: differing effects of uncertainty between an increase in risk and that in Knightian uncertainty. Appendix A contains derivations of important formulae in Section 3. The concept of rectangularity of a set of density generators[3], of which the κ-ignorance is a special case, plays an important role in the following analysis. Appendix B provides some results on rectangularity for readers' convenience and for our exposition to be self-contained.

2. A Two-period, Two-state Illustrative Example

This section offers an illustrating example of differing effects between risk and Knightian uncertainty. The example is a simple patent-pricing one and essentially the same as the widget-factory example of Dixit and Pindyck (1994, Chapter 2). We compare the effect of an increase in risk with that of an increase in Knightian uncertainty, on the value of a patent. We show an increase in Knightian uncertainty reduces the value of the patent, while an increase in risk increases its value. In contrast, both have the same effect on the value of waiting: they make waiting more likely.

Consider a risk-neutral firm contemplating whether or not to buy a patent (or a venture firm, a vacant lot, and so on). After purchasing the patent, the firm has to spend a large amount of money to utilize the patent. The firm may have to build a new factory to produce patented

[3]Theseconcepts are developed and discussed in Chen and Epstein (2001).

products. The factory is product-specific and cannot be used for other purposes. Thus, the investment is irreversible and becomes sunk afterward.

Suppose that there are two periods, period 0 and period 1. There is no uncertainty in period 0 and the operating profit from producing and selling the products is π_0. There is uncertainty in period 1, where the state is either *boom*(b) or *slump*(s). Let π_1 be the operating profit in period 1, which equals π_b in the boom while it equals π_s in the slump.

In order to utilize this patent, the firm has to build a factory to produce the product. Let I be the cost of building the factory. We assume that $\pi_s < I < \pi_b$. The firm has a choice between building the factory in period 0 and in period 1. Let p_b be the probability of boom in period 1 and let r be the rate of interest. Then, if the firm build a factory in period 0, the expected discounted cash flow from this patent is

$$\left(\pi_0 - I\right) + \frac{1}{1+r}\left(p_b \pi_b + \left(1-p_b\right)\pi_s\right), \tag{1}$$

while if it postpones investment until period 1, the expected discounted cash flow of this patent is

$$\frac{p_b}{1+r}\left(\pi_b - I\right), \tag{2}$$

since the period 1's cash flow is $\pi_b - I$ in boom and 0 in slump (that is, the firm does not want to build the factory in slump since $\pi_s < I$).

If the firm is perfectly certain that the boom probability is p_b, the model is exactly the same as the Dixit-Pindyck example. The firm compares (1) with (2) and determines the optimal timing of the investment. Then, the value of the unutilized patent is determined accordingly. Thus, if

$$\pi_0 - \left(1 - \frac{p_b}{1+r}\right) I + \frac{1}{1+r}\left(1-p_b\right)\pi_s < 0 \tag{3}$$

holds, then postponement is the optimal strategy. If otherwise, to invest in period 0 is optimal. Consequently, if (3) holds true (that is, postponement is optimal), then (2) is the value of the patent at period 0. If otherwise, (1) is the value of the patent at period 0.

In the real world, however, it is not likely that the firm is absolutely certain about the boom probability. The firm may think that p_b is likely to be the boom probability, but at the same time it may consider that another probability, say, p'_b, is also plausible. Moreover, the firm may not be at all certain whether a particular boom probability is "more plausible" than others. In sum, the firm may have a set of boom probabilities, instead of having one boom probability as in the Dixit-Pindyck example. Moreover, the firm may not be certain about "relative plausibility" of these boom probabilities. Such a multiplicity of probability distributions is called *Knightian uncertainty*.

Let $\mathcal{P}$ be a compact set of boom probabilities that the firm thinks plausible. It is known (see Gilboa and Schmeidler, 1989) that in multiple-probability-measure cases of this kind, if the firm's behavior is in accordance with certain sensible axioms, then the firm's behavior is characterized as being *uncertainty-averse*: when the firm evaluates its position, it uses a probability corresponding to the "worst" scenario. This means (1) is replaced by

$$\begin{aligned} & \left(\pi_0 - I\right) + \frac{1}{1+r} \min_{p_b \in \mathcal{P}} \left(p_b \pi_b + \left(1 - p_b\right) \pi_s\right) \\ = \quad & \left(\pi_0 - I\right) + \frac{1}{1+r} \left[\pi_s + \left(\min_{p_b \in \mathcal{P}} p_b\right) \left(\pi_b - \pi_s\right)\right] \end{aligned} \tag{4}$$

since $\pi_b > \pi_s$, and (2) is now

$$\frac{1}{1+r} \min_{p_b \in \mathcal{P}} p_b \left(\pi_b - I\right) = \frac{1}{1+r} \left[\left(\min_{p_b \in \mathcal{P}} p_b\right) \left(\pi_b - I\right)\right] \tag{5}$$

since $\pi_b > I$. Consequently, the postponement criterion is now

$$\left(\pi_0 - I\right) + \frac{\pi_s}{1+r} + \frac{1}{1+r} \left(\min_{p_b \in \mathcal{P}} p_b\right) \left(I - \pi_s\right) < 0\,. \tag{6}$$

Let us now consider an increase in risk and uncertainty, one by one. To make exposition simple, let us further assume $p_b = 1/2$ and $\mathcal{P} = [(1/2) - \varepsilon, (1/2) + \varepsilon]$. Here, $\varepsilon \in (0, 1/2)$ is a real number which can be described as the degree of "contamination" of confidence in $p_b = 1/2$. We hereafter call this specification the *ε-contamination*.[4] An increase in ε can be considered as an increase in Knightian uncertainty.[5]

[4]The concept of ε-contamination can be applied to multiple-state cases. Let $\mathcal{M}$ be the set of all probability measures over states, and $P_0 \in \mathcal{M}$. Then ε-contamination of P_0, $\{P_0\}^{\varepsilon}$, is defined by:

$$\{P_0\}^{\varepsilon} \equiv \{\,(1-\varepsilon)P_0 + \varepsilon Q \,|\, Q \in \mathcal{M}\,\},$$

where $\varepsilon \in [0, 1)$. It can also be generalized to discrete-time dynamic models. See Nishimura and Ozaki (2001).

[5]For a behavioral foundation of this, see Ghirardato and Marinacci (2002). Learning under the ε-contamination Knightian uncertainty is explored by Nishimura and Ozaki (2002).

An increase in risk is characterized by a mean-preserving spread of the second-period operating profit π_1. Suppose that there is no Knightian uncertainty, and that risk is increased so that (π_s, π_b) is now spread to $(\pi_s - \gamma, \pi_b + \gamma)$. It is evident from (2) that the mean-preserving spread always increases the value of *un*utilized patent, that is the value of patent when the firm postpones investment. At the same time, it leaves the value of the utilized patent unchanged (see (1)). Consequently, the value of investment opportunity increases by an increase in risk.

Furthermore, (1) and (2) shows that the firm is more likely to find it profitable to postpone investment than before when the mean-preserving spread takes place. This is also clear from (3). Its left-hand side decreases by the mean-preserving spread and hence the criterion for the postponement is easier to be satisfied.

Next, suppose that there is Knightian uncertainty, and thus the value of the utilized patent is (4) and that of the unutilized patent is (5). Suppose further that Knightian uncertainty is increased in the sense that the degree of the confidence contamination, ε, is increased. It is straightforward from (4) and (5) to see that an increase in ε always decreases the value of both the utilized patent and the unutilized one. Therefore, the value of investment opportunity *de*creases, rather than increases, by an increase in Knightian uncertainty.

It is also evident from (4) and (5) that the reduction in value is larger in the utilized patent than in unutilized patent, implying that the firm is again more likely to postpone investment than before. This is also clear from (6). Its left-hand side decreases by an increase in ε and hence the criterion for the postponement is easier to be satisfied.

Thus, an increase in Knightian uncertainty and an increase in risk have opposite effects on the value of the unutilized patent (the investment opportunity) although they have the same effects on the timing of investment. In the following sections, we argue the same result holds in general continuous-time models.

3. The Value of A Patent under Knightian Uncertainty

In this section, the simple two-period two-state example is generalized to a continuous-time model. In Section 3.1 the model is set up, and the conventional model of no Knightian uncertainty is reviewed. In Section 3.2, Knightian uncertainty is introduced into this model.

The value of a utilized patent is derived in Section 3.3. The value of an unutilized patent is examined in Sections 3.4 and 3.5.

3.1. A General Continuous-Time Model

Let T be the expiration date of the patent, which is assumed to be finite for time being. For simplicity, we assume the patent produces no profit after its expiration date T.[6] Later, we consider the case in which T is infinite.

In the conventional model, there is no Knightian uncertainty and the firm is perfectly certain about a probability measure governing the operating profit from the utilized patent. Let $(\Omega, \mathscr{F}_T, P)$ be a probability space and let $(B_t)_{0 \le t \le T}$ be a standard Brownian motion with respect to P. As a filtration $(\mathscr{F}_t)_{0 \le t \le T}$, we take the *standard filtration* for (B_t).[7] The operating profit from the utilized patent is a real-valued stochastic process $(\pi_t)_{0 \le t \le T}$ that is generated by a geometric Brownian motion such that

$$d\pi_t = \mu \pi_t dt + \sigma \pi_t dB_t \tag{7}$$

with $\pi_0 > 0$. Here, μ and σ are some real numbers. We assume that $\sigma \ge 0$ without loss of generality[8] and $\sigma \ne 0$ to exclude a deterministic case.

The value at time t of the utilized patent with expiration time T, which has the current profit flow π_t, is defined by

$$W(\pi_t, t) = E^P \left[\int_t^T e^{-\rho(s-t)} \pi_s \, ds \,\middle|\, \mathscr{F}_t \right], \tag{8}$$

where $\rho > 0$ is the firm's discount rate and $E^P[\cdot|\mathscr{F}_t]$ denotes the expectation with respect to P conditional on $\mathscr{F}_t$. We assume $\rho > \mu$.

The firm's problem is to determine the timing of incurring a cost of investment I. The optimal time is the solution to the optimal stopping problem of finding an $(\mathscr{F}_t)$-stopping time[9], t' ($t' \in [0, T]$), that maximizes the value of the patent at period 0 :

$$E^P \left[\int_{t'}^T e^{-\rho s} \pi_s \, ds - e^{-\rho t'} I \,\middle|\, \mathscr{F}_0 \right]. \tag{9}$$

[6] To incorporate a possibility of after-expiration profit flow is straightforward but makes analysis cumbersome.

[7] A filtration $(\mathscr{F}_t)$ is the *standard filtration* for (B_t) if for each $t \ge 0$, $\mathscr{F}_t$ is the smallest σ-algebra that contains all P-null sets and with respect to that $(B_k)_{0 \le k \le t}$ are all measurable.

[8] If necessary, take $(-B_t)$ instead of (B_t).

[9] That is, t' is such that $(\forall t \ge 0)\ \{t' \le t\} \in \mathscr{F}_t$.

Consequently, the value at time t of the unutilized patent, V_t, is defined by

$$V_t = \max_{t' \in [t,T]} E^P \left[\int_{t'}^{T} e^{-\rho(s-t)} \pi_s \, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathscr{F}_t \right] .$$

Since the analysis of this model is a standard one, we only sketch it here.[10] It is straightforward to show that $W(\pi_t, t)$ is such that

$$W(\pi_t, t) = \int_t^T \pi_t \exp\left(-(\rho - \mu)(s-t)\right) ds = \frac{\pi_t}{\rho - \mu} \left(1 - e^{-(\rho-\mu)(T-t)}\right) . \tag{10}$$

It is customarily assumed for simplicity that the planning horizon is infinite and that the patent never expires, and we follow this lead. Then, the model becomes stationary. In particular, the value of the utilized patent $W(\pi_t, t)$ is equal to

$$W(\pi_t) = \frac{\pi_t}{\rho - \mu}, \tag{11}$$

which does not depend directly on t. Applying Ito's lemma to (11), we have

$$dW_t = \mu W_t dt + \sigma W_t dB_t$$

with $W_0 = \pi_0 (\rho - \mu)^{-1}$.

Since the model is stationary, the value of patent V_t is shown to depend only on W_t and satisfies the following Hamilton-Jacobi-Bellman functional equation:

$$V(W_t) = \max\left\{ W_t - I, E^P\left[\, dV_t \mid \mathscr{F}_t \right] + V(W_t) - \rho V(W_t)\, dt \right\} .$$

The optimal strategy is shown to be a reservation one (that is, to stop and invest now if $W_t \geq W^*$ and to continue if otherwise, where W^* is the reservation value), and we have from the above equation,

$$E^P\left[\, dV_t \mid \mathscr{F}_t \right] = \rho V(W_t) dt$$

in the continuation region (that is, if $W_t > W^*$). Applying Ito's lemma on $V(W_t)$ and substituting the result into the above equation, we have a non-stochastic second-order ordinary differential equation of $V(W_t)$ such that

$$\frac{1}{2}\sigma^2 W_t^2 V''(W_t) + \mu W_t V'(W_t) - \rho V(W_t) = 0 ,$$

[10]The following results will be verified later in this section as we examine Knightian uncertainty, since no-Knightian uncertainty is a special case of Knightian uncertainty.

with boundary conditions of $V(0)=0$, $V(W^*)=W^*-I$ (corresponding to the value matching condition) and $V'(W^*)=1$ (corresponding to the smooth pasting condition).

Solving this ordinary differential equation, we have the value of the unutilized patent (that is, the value of the patent in the continuation region) such that

$$V(W_t)=\left(\frac{I}{\alpha-1}\right)^{1-\alpha}\alpha^{-\alpha}W_t^{\alpha} \tag{12}$$

and the reservation W^* such that

$$W^*=\frac{\alpha I}{\alpha-1}, \tag{13}$$

where α is a constant defined by

$$\alpha=\frac{-\left(\mu-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}}{\sigma^2}. \tag{14}$$

3.2. Knightian Uncertainty in Continuous Time

As in the simple example of Section 2, it is not likely that the firm has perfect confidence about the probability measure P. There may be other candidate probability measures that the firm considers probable. Thus, the firm faces Knightian uncertainty, in which the firm confronts a set of probability measures instead of a single probability measure P. However, these candidate probability measures are not likely to be wildly different from P, but it is rather a small deviation from P, like ε-contamination in the example of the previous section.

To model this type of Knightian uncertainty in the continuous-time framework, we follow Chen and Epstein (2001) who characterized Knightian uncertainty in continuous time in a different context. Firstly, we assume that the firm considers only a set of probability measures that have perfect agreement with P with respect to zero probability events. This amounts to assume that probability measures we consider is absolutely continuous with respect to P and one another. Two probability measures are called equivalent if they are absolutely continuous with each other. Thus, we are concerned with probability measures equivalent to P.[11]

[11]This is a weaker concept of Knightian uncertainty than that in the discrete-time model (see, for example, Nishimura and Ozaki, 2001).

It is known (Girsanov's Theorem) in the mathematical finance literature that an equivalent measure is generated by a "density generator" from an original probability measure, and that a Brownian motion (with respect to the original probability measure) perturbed by this density generator is still a Brownian motion (with respect to the generated equivalent measure). Moreover, all equivalent measures to P can be generated in this way. This property in turn implies that "geometric Brownian motions" corresponding to probability measures equivalent to P, which describe the movement of profit π_t under these probability measures, are different only in their drift term, not in the volatility term.

Secondly, we assume that the firm considers only small perturbations from P: that is, the density generator is confined in a small range. In particular, we are mostly concerned with a continuous counterpart of ε-contamination in Section 2, which Chen and Epstein call κ-ignorance. Specifically, κ-ignorance assumes that the density generator moves only in the range $[-\kappa,\kappa]$ and thus κ can be considered as the degree of Knightian uncertainty or ignorance.

Under this formulation, we show an argument similar to that in the conventional case (sketched in Section 3.1) holds true with appropriate changes in variables even under Knightian uncertainty of this type. In particular, we have a variant of (10), (12), (13), and (14). This representation is utilized in the sensitivity analysis of the next section.

3.2.1. Density Generators, Girsanov's Theorem and Characterization of Knightian Uncertainty

Let us now define "density generators" and their properties that are utilized in formulating Knightian uncertainty in this paper. Let $\mathscr{L}$ be the set of real-valued, measurable[12], and $(\mathscr{F}_t)$-adapted stochastic processes on $(\Omega,\mathscr{F}_T,P)$ with an index set $[0,T]$ and let $\mathscr{L}^2$ be a subset of $\mathscr{L}$ which is defined by

$$\mathscr{L}^2=\left\{(\theta_t)_{0\le t\le T}\in\mathscr{L}\,\middle|\,\int_0^T\theta_t^2\,dt<+\infty\quad P\text{-a.s.}\right\}.$$

Given $\theta=(\theta_t)\in\mathscr{L}^2$, define a stochastic process $\left(z_t^\theta\right)_{0\le t\le T}$ by

$$(\forall t)\quad z_t^\theta=\exp\left(-\frac{1}{2}\int_0^t\theta_s^2\,ds-\int_0^t\theta_s dB_s\right),\tag{15}$$

[12]A real-valued stochastic process $(X_t)_{0\le t\le T}$ on $(\Omega,\mathscr{F}_T,P)$ is *measurable* if a function $(t,\omega)\mapsto X_t(\omega)$ is $(\mathscr{B}([0,T])\otimes\mathscr{F}_T)$-measurable, where $\mathscr{B}([0,T])$ is the Borel σ-algebra on $[0,T]$.

where the stochastic integral, $\int_0^t \theta_s dB_s$, is well-defined for each t since $\theta \in \mathscr{L}^2$.[13] A stochastic process $(\theta_t) \in \mathscr{L}^2$ is called a *density generator* if $\left(z_t^\theta\right)$ thus defined is a $(\mathscr{F}_t)$-martingale. A sufficient condition for $\left(z_t^\theta\right)$ to be $(\mathscr{F}_t)$-martingale and thus for (θ_t) to be a density generator is *Novikov's condition*:

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2\, ds\right)\right] < +\infty \tag{16}$$

(see Karatzas and Shreve, 1991, p.199, Corollary 5.13).

As the name suggests, a density generator generates another probability measure from a given probability measure, and the resulting measure is equivalent to the original measure. To see this, let θ be a density generator and define the probability measure Q^θ by

$$(\forall A \in \mathscr{F}_T) \quad Q^\theta(A) = \int_A z_T^\theta(\omega) dP(\omega)\,. \tag{17}$$

Since $\left(z_t^\theta\right)$ is a martingale, $Q^\theta(\Omega) = E^P[z_T^\theta] = z_0^\theta = 1$, and hence, Q^θ is certainly a probability measure and it is absolutely continuous with respect to P. Furthermore, since z_T^θ is strictly positive, P is absolutely continuous with respect to Q^θ. Therefore, Q^θ is equivalent to P. Conversely, any probability measure which is equivalent to P can be generated via (17) by some density generator (see Duffie, 1996, p.289).

We assume that the firm's set of probability measures describing its Knightian uncertainty consists of probability measures equivalent to P. The standard results described in the previous paragraph then imply that the firm's Knightian uncertainty is characterized as an "expansion" of the set of probability measures from a singleton set $\{P\}$ through density generators.

Let Θ be a set of density generators. We then define the set of probability measures generated by Θ, $\mathcal{P}^\Theta$, on $(\Omega, \mathscr{F}_T)$ by

$$\mathcal{P}^\Theta = \{\, Q^\theta \mid \theta \in \Theta \,\}\,, \tag{18}$$

where Q^θ is derived from P according to (17). Thus, the firm's Knightian uncertainty is characterized by $\mathcal{P}^\Theta$ for some Θ.

[13]Equivalently, $\left(z_t^\theta\right)_{0\le t\le T}$ is defined as a unique solution to the stochastic differential equation: $dz_t^\theta = -z_t^\theta \theta_t dB_t$ with $z_0^\theta = 1$.

This specification has a strong implication: if we define, for each $\theta \in \Theta$, a stochastic process $\left(B_t^\theta\right)_{0 \le t \le T}$ by

$$(\forall t) \quad B_t^\theta = B_t + \int_0^t \theta_s \, ds\,, \tag{19}$$

then $\left(B_t^\theta\right)$ turns out to be a standard Brownian motion with respect to $(\mathscr{F}_t)$ on $(\Omega, \mathscr{F}_T, Q^\theta)$ (Girsanov's theorem: Karatzas and Shreve, 1991, p.191, Theorem 5.1).

3.2.2. A Corresponding Set of Stochastic Differential Equations

Recall that the real-valued stochastic process $(\pi_t)_{0 \le t \le T}$ is generated by a stochastic differential equation such that

$$d\pi_t = \mu \pi_t dt + \sigma \pi_t dB_t \tag{20}$$

with $\pi_0 > 0$, where (B_t) is a Brownian motion with respect to the probability measure P. Since $dB_t^\theta = dB_t + \theta_t dt$ from (19), we have for any $\theta \in \Theta$,

$$d\pi_t = (\mu - \sigma\theta_t)\pi_t dt + \sigma \pi_t dB_t^\theta\,. \tag{21}$$

Thus, (π_t) is also the solution of the stochastic differential equation (21) if Q^θ is the probability measure, because in this case, $\left(B_t^\theta\right)$ is a Brownian motion with respect to Q^θ by Girsanov's theorem.

Under uncertainty characterized by Θ, the decision-maker considers all stochastic differential equations, (21), with $\theta \in \Theta$ varying. It should be noted here that there exists only one stochastic process (π_t) and this process has many "interpretations" of stochastic differential equations. One such interpretation is (20) and other ones are (21), each of which corresponds to each $\theta \in \Theta$. It should also be noted that θ affects only the drift term, not the volatility term, in (21).

Let $\theta \in \Theta$. Then, by (21) and an application of Ito's lemma to the logarithm of π_t by regarding Q^θ as the true probability measure, we obtain

$$(\forall t \ge 0) \quad \pi_t = \pi_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t - \sigma \int_0^t \theta_s \, ds + \sigma B_t^\theta\right)\,. \tag{22}$$

This clearly shows that a single stochastic process (π_t) has many distinct interpretations, each of which corresponds to a particular probability measure equivalent to P.

3.2.3. Strong Rectangularity, i.i.d. Uncertainty and κ-ignorance

We assume that the firm considers "small deviation" from the original P. This means that the range that density generators can move is restricted to some "neighborhood" set of P. In particular, we consider a counterpart of ε-contamination of Section 2, which is called κ-ignorance (Chen and Epstein, 2001).

In this section, we formally define κ-ignorance. However, since some of our results in this paper hold true under weaker conditions than κ-ignorance, it is worthwhile to define these conditions: rectangularity and *i.i.d.* uncertainty in the terminology of Chen and Epstein.

A set of density generators, Θ, is called *strongly rectangular* if there exist a nonempty compact subset $\mathscr{K}$ of $\mathbb{R}$ and a compact-valued, measurable[14] correspondence $K : [0,T] \twoheadrightarrow \mathscr{K}$ such that

$$\Theta = \left\{ (\theta_t) \in \mathscr{L}^2 \,\middle|\, \theta_t(\omega) \in K_t \ \ (m \otimes P)\text{-a.s.} \right\}, \tag{23}$$

where m denotes the Lebesgue measure restricted on $\mathscr{B}([0,T])$.[15] Any element of a set Θ defined by (23) satisfies Novikov's condition (16) since $(\forall t)\, K_t \subseteq \mathscr{K}$ and $\mathscr{K}$ is compact, and hence, it is certainly a density generator. We denote by $\Theta^{(K_t)}$ the set defined by (23).

To see an important implication of strong rectangularity, let $0 \le s \le t \le T$, let x be an $\mathscr{F}_T$-measurable random variable and let Θ be a strongly rectangular set of density generators. Then, it holds that

$$\min_{\theta\in\Theta} E^{Q^\theta}[x \mid \mathscr{F}_s] = \min_{\theta\in\Theta} E^{Q^\theta}\left[E^{Q^\theta}[x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] = \min_{\theta\in\Theta} E^{Q^\theta}\left[\min_{\theta'\in\Theta} E^{Q^{\theta'}}[x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] \tag{24}$$

[14] A compact-valued correspondence $K : [0,T] \twoheadrightarrow \mathscr{K}$ is *measurable* by definition if $\{t \in [0,T] \mid K_t \cap U \neq \phi\} \in \mathscr{B}([0,T])$ holds for any open set U.

[15] Chen and Epstein (2001) adopt a definition of rectangularity which requires less than ours. That is, a set of density generators, Θ, is *rectangular* if there exists a set-valued stochastic process $(K_t)_{0\le t\le T}$ such that

$$\Theta = \left\{ (\theta_t) \in \mathscr{L}^2 \,\middle|\, \theta_t(\omega) \in K_t(\omega) \ \ (m \otimes P)\text{-a.s.} \right\}$$

and, for each t, $K_t : \Omega \twoheadrightarrow \mathbb{R}$ is compact-valued and satisfies some additional regularity conditions. Our definition of strong rectangularity further restricts K_t to be degenerate (that is, non-stochastic). This further restrction makes the following analysis much easier.

as long as the minima exist (Lemma B3 in the Appendix B). Note that while the first equality always holds by the law of iterated integrals, the second equality may not hold if Θ is not strongly rectangular.[16] The "recursive" structure (24) under strong rectangularity is exploited when we solve an optimal stopping problem for the firm.

The *i.i.d.* uncertainty is a special case of strong rectangularity, in which K_t is independent of time t. To be precise, the uncertainty characterized by Θ is said to be an *i.i.d. uncertainty*, if there exists a compact subset K of $\mathbb{R}$ such that $0 \in K$ and

$$\Theta = \left\{ \, (\theta_t) \in \mathscr{L}^2 \,\middle|\, \theta_t(\omega) \in K \;\; (m \otimes P)\text{-a.s.} \right\} . \tag{25}$$

We denote Θ in (25) by Θ^K.

Finally, the κ-*ignorance* is a special case of *i.i.d.* uncertainty, where the set K is further specified as

$$K = [-\kappa, \kappa]$$

for some $\kappa \geq 0$. It is evident that if $\kappa = 0$, the Knightian uncertainty vanishes. If κ increases, this means that the firm is less certain than before that candidate probability measures are close to P. Thus, κ-ignorance can be considered as a continuous counterpart of ε-contamination in Section 2.

3.3. The Value of A Utilized Patent under Strong Rectangularity

Let us now consider the value of a utilized patent under Knightian uncertainty described in the previous subsection. We first derive its exact formula. Here, it turns out that strong rectangularity, of which *i.i.d.* uncertainty (and thus κ-ignorance) is a special case, is sufficient to obtain a simple formula comparable to the one under no Knightian uncertainty.

Suppose that $\mathcal{P}^\Theta$ is Knightian uncertainty that the firm faces, where $\mathcal{P}^\Theta$ is the set of measures defined by (18) with some strongly rectangular set of density generators, $\Theta = \Theta^{(K_t)}$. Then, the corresponding profit process (π_t) follows (21). As in Section 2 we assume the firm

[16] Note that the second equality in (24) also holds under the weaker requirement of rectangularity of Chen and Epstein (2001) (see the previous footnote). Strong rectangularity is needed to show Proposition 1 below, where the weaker form of rectangularity is not sufficient.

is uncertainty-averse. Then, the value at time t of the utilized patent with expiration time T, which has the current profit flow π_t, is defined by

$$W(\pi_t, t) = \inf_{Q \in \mathcal{P}^{\Theta}} E^Q \left[\int_t^T e^{-\rho(s-t)} \pi_s \, ds \,\middle|\, \mathscr{F}_t \right], \tag{26}$$

where $\rho > 0$ is the firm's discount rate and $E^Q[\cdot|\mathscr{F}_t]$ denotes the expectation with respect to Q conditional on $\mathscr{F}_t$. The infimum operator reflects the firm's uncertainty aversion.

Before presenting Proposition 1 that gives an exact formula, we need to define an "upper-rim density generator," which plays a pivotal role in our analysis. Given (K_t), define an upper-rim density generator, (θ_t^*), by

$$(\forall t) \quad \theta_t^* \equiv \arg\max \{ \sigma x \,|\, x \in K_t \} = \max K_t, \tag{27}$$

where the equality holds by $\sigma > 0$ and the compact-valuedness of K. Note that we write $\max K_t$, instead of $\{\max K_t\}$. Then, (θ_t^*) turns out to be a degenerate (that is, non-stochastic) measurable process[17], and hence, $(\theta_t^*) \in \mathscr{L}$. Therefore, it follows that $(\theta_t^*) \in \Theta^{(K_t)}$ by (27). Obviously, if Knightian uncertainty is κ-ignorance so that $K_t = [-\kappa, \kappa]$, then $\theta_t^* = \kappa$.

We are now ready to present the exact formula. The proof is relegated to the Appendix A (A.1).

Proposition 1. *Suppose that the firm faces Knightian uncertainty characterized by $\Theta^{(K_t)}$, where $\Theta^{(K_t)}$ is a strongly rectangular set of density generators which is defined by (23) for some (K_t). Then, the value of the utilized patent (26) is given by*

$$W(\pi_t, t) = \int_t^T \pi_t \exp\left(-(\rho - \mu)(s-t) - \int_t^s \sigma \theta_r^* \, dr \right) ds, \tag{28}$$

where (θ_t^*) is defined by (27).

This proposition shows that the value of the utilized patent (28) under Knightian uncertainty with strong rectangularity has an expression comparable to the one under no Knightian uncertainty (10). In addition, we notice that even though the firm is assumed to be risk neutral, the risk factor σ sneaks in and an increase in risk also influence the value of the unutilized

[17] Let $a \in \mathbb{R}$. Then, $\{ t \,|\, \max K_t > a \} = \{ t \,|\, K_t \cap (a, +\infty) \neq \phi \} \in \mathscr{B}([0,T])$ by measurability of K (see the Footnote 13), which shows that $\max K_t$ is $\mathscr{B}([0,T])$-measurable.

patent under Knightian uncertainty, whereas the risk factor does not influence the value under no Knightian uncertainty. Thus, Knightian uncertainty in the continuous-time framework makes the value of the utilized patent depend on not only Knightian uncertainty itself but also risk.

3.4. The Optimal Investment Decision under Strong Rectangularity

In this section, we formulate the investment problem of the firm as an optimal stopping problem under Knightian uncertainty, and relates the investment problem to the value of the utilized patent described in the previous section.

Consider an investment of building a factory to produce patented products, which costs I and in return generates a profit flow $(\pi_s)_{s \geq t}$ when it is made at time t. The firm faces the same Knightian uncertainty as in the previous section: $(\pi_s)_{s \geq t}$ follows (21) with strongly rectangular $\Theta = \Theta^{(K_t)}$ as in the previous section. The firm possessing the patent has this investment opportunity and contemplates the optimal timing of this investment.

Then, at time t, the firm is facing the optimal stopping problem of maximizing

$$\min_{Q \in \mathcal{P}^{\Theta}} E^Q \left[\int_{t'}^{T} e^{-\rho(s-t)} \pi_s \, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathcal{F}_t \right]$$

by choosing an $(\mathcal{F}_t)$-stopping time, t' $(t' \in [t, T])$, when the investment is made. The maximum of this problem is denoted by V_t :

$$V_t = \max_{t' \geq t} \min_{Q \in \mathcal{P}^{\Theta}} E^Q \left[\int_{t'}^{T} e^{-\rho(s-t)} \pi_s \, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathcal{F}_t \right] \tag{29}$$

Then, V_t is the value of the investment opportunity.

Now consider the two options available to the firm: to invest now (that is, at time t) or to wait for a short time interval, dt, and reconsider whether to invest or not after that (that is, at time $t + dt$). Then, as A.2 in the Appendix A shows, V_t solves a version of the *Hamilton-Jacobi-Bellman equation*:

$$V_t = \max \left\{ W_t - I, \min_{Q \in \mathcal{P}^{\Theta}} E^Q [\, dV_t \mid \mathcal{F}_t \,] + V_t - \rho V_t \, dt \right\} . \tag{30}$$

Here, the first term in the right-hand side is the value of “to stop right now” and the second term is the value of “to wait,” each of which corresponds to one of the two options mentioned above.

In general, it is difficult to derive an analytic solution of this functional equation and to get a simple formula of the value of the unutilized patent. However, analysis is greatly simplified if (a) underlying Knightian uncertainty is further restricted to *i.i.d.* uncertainty, (b) the planning horizon is infinite and (c) the patent never expires. In the next section, we explicitly solve the optimal stopping problem in that case and get a simple pricing formula of the unutilized patent analogous to the one under no Knightian uncertainty.

3.5. The Value of An Unutilized Patent under *i.i.d.* Uncertainty and Infinite Horizon

3.5.1. The i.i.d. Uncertainty and Infinite Horizon

Under *i.i.d.* uncertainty, the upper-rim density generator θ_t^* defined by (27) is independent of t and given by

$$\theta^* = \arg\max\{\, \sigma x \,|\, x \in K \,\} = \max K\,. \tag{31}$$

Therefore, (28) is simplified to

$$W(\pi_t, t) = \int_t^T \pi_t e^{(-\rho+\mu-\sigma\theta^*)(s-t)}\, ds = \frac{\pi_t}{\lambda}\left(1 - e^{-\lambda(T-t)}\right) \tag{32}$$

where $\lambda \in \mathbb{R}$ is defined by

$$\lambda \equiv \rho - (\mu - \sigma\theta^*)\,. \tag{33}$$

In this section, we consider the case that (a) the risk-neutral firm faces *i.i.d.* uncertainty, (b) the planning horizon of this firm is infinite and (c) the expiration date of the patent is also infinite. (In the terminology of option theory, we are now considering irreversible investment as an American option.)

In what follows, we let T go to infinity, and assume that relations between variables in the limit hold true in the infinite horizon case. We have assumed that $\rho > \mu$, and then, λ defined by (33) turns out to be positive since $0 \in K$. Consequently, under the *i.i.d.* uncertainty, when T approaches infinity, (32) approaches

$$W(\pi_t) = \frac{\pi_t}{\lambda}\,, \tag{34}$$

which is independent of time. We hereafter assume that (34) holds in the infinite time case.[18]

Then, applying Ito's lemma by regarding P as the true probability measure, we have that (W_t) solves

$$dW_t = \mu W_t dt + \sigma W_t dB_t \tag{35}$$

with $W_0 = \pi_0/\lambda$. Hence, (19) implies that for any $\theta \in \Theta$, (W_t) solves

$$dW_t = (\mu - \sigma\theta_t)W_t dt + \sigma W_t dB_t^\theta \,. \tag{36}$$

This shows that when Q^θ is the true probability measure, (W_t) solves the stochastic differential equation defined by (36).

3.5.2. Time-homogeneous Hamilton-Jacobi-Bellman Equation

Let us now come back to the optimal stopping problem of the previous subsection. If the planning horizon is infinite and (W_t) follows (35) (and (36)), then V_t defined by (29) depends only on W_t, not on the physical time t. Therefore, we are allowed to write as $V_t = V(W_t)$ with some $V : \mathbb{R}_+ \to \mathbb{R}$. In this case, the Hamilton-Jacobi-Bellman equation turns out to be

$$V(W_t) = \max\left\{ W_t - I,\ \min_{Q\in\mathcal{P}^\Theta} E^Q[\, dV_t \mid \mathscr{F}_t] + V(W_t) - \rho V(W_t)\, dt \right\}. \tag{37}$$

Let us now solve the above Hamilton-Jacobi-Bellman equation. We conjecture that there exists W^* such that the optimal strategy of the firm takes the form: "stop right now if $W_t \geq W^*$ and wait if $W_t < W^*$." This conjecture shall be verified to be true later.

In the continuation region, that is, when $W_t < W^*$, it holds from (37) that

$$\min_{Q\in\mathcal{P}^\Theta} E^Q[\, dV_t \mid \mathscr{F}_t] = \rho V(W_t)\, dt\,. \tag{38}$$

Here, the left-hand side is the minimum of the capital gain of holding the investment opportunity during $[t, t+dt]$ and the right-hand side is its opportunity cost measured in terms of the firm's discount rate. Equation (38) shows that both must be equal in the continuation region.

[18] When the plannng horizon and the patent expiration date are infinite with the *i.i.d.* uncertainty, W becomes independent of t. However, conditions guaranteeing that (34) holds in the infinite horizon case are not yet known. Here we do not attempt to derive these conditions, and instead we simply assume that such conditions are satisfied in our model. This is because we need more mathematical aparatus than we have in this paper. For example, Girsanov's theorem is usually stated in the finite-horizon framework and thus some sophistication will be required to extend it to an infinite horizon (see Karatzas and Shreve, 1996, p.192, Corollary 5.2).

We now derive from equation (38) a non-stochastic ordinary differential equation of V. Conjecture that V is twice differentiable in the continuation region. Then, by Ito's lemma and (36), we have for each $\theta \in \Theta^K$,

$$dV_t = V'(W_t)\left((\mu - \sigma\theta_t)W_t dt + \sigma W_t dB_t^\theta\right) + \frac{1}{2}\sigma^2 W_t^2 V''(W_t)\,dt\,. \tag{39}$$

At this stage, we conjecture that V' is positive. Then, the left-hand side of (38) is further rewritten (see A.3 in the Appendix A) as

$$\min_{Q\in\mathcal{P}^\Theta} E^Q[\,dV_t \mid \mathcal{F}_t\,] = V'(W_t)\,(\mu - \sigma\theta^*)W_t dt + \frac{1}{2}\sigma^2 W_t^2 V''(W_t)\,dt\,, \tag{40}$$

where θ^* is defined in (31). From (38) and (40), we obtain a (non-stochastic) second-order ordinary differential equation of V such that

$$\frac{1}{2}\sigma^2 W_t^2 V''(W_t) + (\mu - \sigma\theta^*)W_t V'(W_t) - \rho V(W_t) = 0\,, \tag{41}$$

which must hold in the continuation region.

In order to solve (41) for V, we need two boundary conditions. One boundary condition is given by the condition that if the utilized patent has no value then the investment opportunity is also of no value[19]:

$$V(0) = 0\,. \tag{42}$$

The other boundary condition follows from the Hamilton-Jacobi-Bellman equation (37) as

$$V(W^*) = W^* - I\,, \tag{43}$$

where W^* is the "reservation value" whose existence is now being assumed.

Since W^* must be chosen optimally by the firm, (43) serves as a free-boundary condition. In order to determine the value of W^*, we need an additional condition on V, which is obtained from the optimization with respect to W^*. To find this, consider the gain the firm would obtain when it made an investment upon observing W_t. It would be the value of the project minus the

[19]>From (36), it follows that $W_T = 0$ for any $T \geq t$ if $W_t = 0$. Hence, it is optimal for the firm never to invest, leading to (42) by (29).

value of the investment opportunity (that is, the value of not making investment now) and it is given by $W_t - V(W_t)$. Since W^* should be chosen so as to maximize this, it must hold that

$$V'(W^*) = 1 \tag{44}$$

from the first-order condition for the maximization.[20]

3.5.3. The Optimal Strategy

The ordinary differential equation (41) with boundary conditions (42), (43) and (44) can be explicitly solved to obtain (see A.4 in the Appendix A)

$$V(W_t) = \left(\frac{I}{\alpha-1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^{\alpha} \tag{45}$$

as far as $W_t < W^*$, where the reservation value W^* is given by

$$W^* = \frac{\alpha I}{\alpha - 1} \tag{46}$$

and α is a constant defined by

$$\alpha = \frac{-\left\{(\mu - \sigma\theta^*) - \frac{1}{2}\sigma^2\right\} + \sqrt{\left\{(\mu - \sigma\theta^*) - \frac{1}{2}\sigma^2\right\}^2 + 2\rho\sigma^2}}{\sigma^2}. \tag{47}$$

Under the maintained assumption that $\rho > \mu$ and $0 \in K$, it holds that $\alpha > 1$ (see A.5 in the Appendix A). Hence, W^* and V are well-defined.

Therefore, the value of the investment opportunity or the patent, V, is given by

$$V(W_t) = \begin{cases} \left(\frac{I}{\alpha-1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^{\alpha} & \text{if} \quad W_t < W^* \\ W_t - I & \text{if} \quad W_t \geq W^* \end{cases} \tag{48}$$

(see Figure 1). Recall that we have made three conjectures: (a) There exists a unique reservation value W^*; (b) V is twicely differentiable in the continuation region; and (c) V' is positive. These conjectures are easily verified to hold true from (48) ((b) and (c) are immediate and see Figure 1 for (a)).

To sum up, we have proved

[20] We will find later that V is convex, and hence, the second-order condition turns out to be satisfied.

Proposition 2. *Assume that Knightian uncertainty the firm faces is i.i.d. uncertainty characterized by* $\Theta = \Theta^K$ *for some K, and assume further that relations among variables in the finite-horizon case converge, as the horizon goes to infinity, to those in the infinite-horizon case. Then, in the case of infinite horizon, the value of the unutilized patent, that is,* $V(W_t)$ *in the continuation region (* $W_t < W^*$*), is given by (45) with* W^* *and* α *defined by (46) and (47), respectively.*

It should be noted that (45), (46) and (47) are comparable with (12), (13), and (14) in the case of no Knightian uncertainty. The only difference is that μ under no Knightian uncertainty is replaced by $\mu - \sigma\theta^*$ when Knightian uncertainty is present in the form of *i.i.d.* one. In fact, no Knightian uncertainty case is the special case that $\theta^* = 0$, and it is straightforward to see that α in the case of no Knightian uncertainty is (14) and the value of the unutilized patent is (12).

4. Sensitivity Analysis

This section compares the effect of an increase in Knightian uncertainty and that of an increase in risk, on the value of a patent, and on the optimal timing of investment. We show the same result holds in the continuous-time case as in the simple two-period two-state example of Section 2.

4.1. An Increase in Risk

Let us consider the effect of an increase in risk when there is no Knightian uncertainty ((11), (12), (13), and (14)). It is evident from (11) that σ^2 does not influence the value of the utilized patent. In contrast, σ^2 changes the value of the unutilized patent. With some calculations, we have

$$\frac{\partial \alpha}{\partial \sigma^2} < 0. \tag{49}$$

We also obtain[21] $\partial V(W_t)/\partial \alpha < 0$ for the unutilized patent (that is, when $W_t < W^*$). Combining these two relations, we have $\partial V(W_t)/\partial \sigma^2 > 0$ for the value of the unutilized patent. Thus, an

[21]The claim holds since

$$\partial \ln V(W_t)/\partial \alpha \quad = \quad -\ln I + \ln(\alpha - 1) - \ln \alpha + \ln W_t$$

increase in σ^2 increases the value of the unutilized patent. Finally, since $\partial W^*/\partial\alpha < 0$ (see (13)), we obtain $\partial W^*/\partial\sigma^2 > 0$ by (49). Therefore, we get

Proposition 3. *In the case of no Knightian uncertainty, an increase in σ^2, that is, an increase in risk, induces (a) an increase in the value of the unutilized patent and no change in the value of the utilized patent, and (b) an increase in the reservation value W^* (see Figure 2).*

4.2. An Increase in Knightian Uncertainty (κ-ignorance)

Let us now consider the case that Knightian uncertainty the firm faces is characterized as κ-ignorance, which is a continuous counterpart of ε-contamination of the two-period two-state example of Section 2.

Under κ-ignorance, the upper-rim density generator θ_t^* defined by (27) is given by

$$\theta^* = \arg\max\{\, \sigma x \,|\, x \in [-\kappa, \kappa]\,\} = \kappa .$$

Then, since κ-ignorance is a special case of the *i.i.d.* uncertainty, we immediately get from (33)

$$\lambda = \rho - (\mu - \kappa\sigma)\,. \tag{50}$$

Consequently, the relevant equations are now (34), (50), (45), (46) and (47).

Consider first the value of the utilized patent. We have from (34) and (50)

$$W_t = W(\pi_t) = \frac{\pi_t}{\rho - (\mu - \kappa\sigma)}\,. \tag{51}$$

Consequently, an increase in κ reduces the value of the utilized patent. It is contrastive to the case of no Knightian uncertainty where an increase in σ^2 has no effect on the value of the utilized patent.

Let us now turn to the value of the unutilized patent. From (45), it is evident that an increase in κ reduces the value of the unutilized patent through lowering W. Moreover, an

---

$$\begin{aligned} &< \;\; -\ln I + \ln(\alpha-1) - \ln\alpha + \ln W^* \\ &= \;\; -\ln I + \ln(\alpha-1) - \ln\alpha + \ln\left(\frac{\alpha I}{\alpha-1}\right) = 0\,. \end{aligned}$$

increase in κ further reduces the value of the unutilized patent through α. We have from (50) and (47)

$$\alpha = \frac{-\left\{(\mu-\kappa\sigma)-\frac{1}{2}\sigma^2\right\}+\sqrt{\left\{(\mu-\kappa\sigma)-\frac{1}{2}\sigma^2\right\}^2+2\rho\sigma^2}}{\sigma^2}. \tag{52}$$

It then follows with some calculations that

$$\frac{\partial\alpha}{\partial\kappa} > 0\,. \tag{53}$$

Thus, the effect of an increase in κ on α is just in the opposite direction to that of an increase in σ^2 under no Knightian uncertainty. Combining this effect and $\partial V(W_t)/\partial\alpha < 0$ when $W_t < W^*$ from (45), we have $(\partial V(W_t)/\partial\alpha)(\partial\alpha/\partial\kappa) < 0$ when $W_t < W^*$. Consequently, an increase in κ reduces the value of the unutilized patent (that is, $V(W_t)$ when $W_t < W^*$) through raising α. The direct effect through lowering W and the indirect effect through raising α both reduces the value of the unutilized patent.

Finally, since $\partial W^*/\partial\alpha < 0$, we obtain $\partial W^*/\partial\kappa < 0$ by (53). Thus, we conclude that the next proposition holds.

Proposition 4. *Assume the same assumptions of Proposition 2. Assume further that the firm's Knightian uncertainty is characterized by* κ*-ignorance. Then an increase in* κ*, that is, an increase in the Knightian uncertainty, induces (a) a* de*crease in the value of the utilized patent and a* de*crease in the value of the unutilized patent, and (b) a* de*crease in the reservation value* W^* *(see Figure 3).*

This proposition provides a sharp contrast with an increase in the "risk" in the previous subsection. While an increase in risk increases the value of the unutilized patent (and leaves the value of the utilized patent unchanged), an increase in Knightian uncertainty (an increase in κ) decreases the value of the unutilized patent (as well as the value of the utilized patent).

4.3. Value of Waiting

Let us now turn to the issue of the value of waiting. To analyze this issue, it is worthwhile to re-interpret the reservation value W^* in Propositions 3 and 4 in terms of the reservation profit flow π^* defined as

$$\pi^* = \begin{cases} (\rho - \mu)W^* & \text{under no Knightian uncertainty} \\ \{\rho - (\mu - \kappa\sigma)\}W^* & \text{under Knightian uncertainty } (\kappa\text{-ignorance}). \end{cases}$$

Note that $W_t = \pi_t/(\rho - \mu)$ under no Knightian uncertainty (see (11)) and $W_t = \pi_t/\{\rho - (\mu - \kappa\sigma)\}$ under κ-ignorance (see (34)). When the current profit flow, π_t, is less than π^*, then the value of utilized patent, W_t, is less than W^* since $\rho - \mu > 0$ under no Knightian uncertainty and $\{\rho - (\mu - \kappa\sigma)\} > 0$ under κ-ignorance. Hence "to wait" is the optimal strategy. To the contrary, when the current profit flow is greater than π^*, then the value of utilized patent is greater than W^* and hence "to stop right now" is the optimal strategy. Therefore, π^* thus defined serves as the reservation profit flow.

It follows that $\partial\pi^*/\partial\sigma^2 > 0$ under no Knightian uncertainty since $\partial W^*/\partial\sigma^2 > 0$ (see Section 4.1) and some calculations show that $\partial\pi^*/\partial\kappa > 0$ under κ-ignorance.[22] Thus, both an increase in risk under no Knightian uncertainty and an increase in κ-ignorance under Knightian uncertainty *in*creases the reservation profit flow, and thus the value of waiting is increased (see Figures 4 and 5). This result conforms to that in Section 2, where an increase both in risk and in Knightian uncertainty makes it more profitable for the firm to postpone irreversible investment.

This waiting-enhancing effect of an increase in Knightian uncertainty is in a stark contrast with its waiting-reducing effect in the job search model investigated by Nishimura and Ozaki (2001). They show in a discrete-time infinite-horizon job search model that an increase in Knightian uncertainty unambiguously reduces the reservation wage and thus shortens waiting.

Although both the job search and irreversible investment models are formulated as an optimal stopping problem, there is a fundamental difference between the two with respect to the nature of uncertainty. In the job search model, the decision-maker determines when she stops job search and thus resolves uncertainty. Thus, an increase in Knightian uncertainty makes the uncertainty-averse decision-maker more likely to stop search and to resolve uncertainty. In contrast, in the irreversible investment model, the decision-maker contemplates when she begins investment and thus faces uncertainty. Thus, an increase in Knightian uncertainty makes the uncertainty-averse decision-maker more likely to postpone investment and to avoid facing

[22] In fact, it can be shown that $\partial\pi^*/\partial\kappa > 0$ if and only if $\alpha > 1$.

uncertainty.

Appendix A: Derivations

A.1. Proof of Proposition 1.

We first show that for any $s \geq t$ and any $\theta \in \Theta^{(K_t)}$,

$$\begin{aligned} & E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r\, dr + \sigma\left(B_s^{\theta} - B_t^{\theta}\right)\right) \Big| \mathscr{F}_t\right] \\ \geq\ & E^{Q^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\, dr + \sigma\left(B_s^{\theta^*} - B_t^{\theta^*}\right)\right) \Big| \mathscr{F}_t\right]. \end{aligned} \tag{54}$$

To do this, let $s \geq t$ and let $\theta \in \Theta^{(K_t)}$. Note that by the definition of (θ_t^*), it holds that

$$(\forall\omega) \quad \exp\left(-\int_t^s \sigma\theta_r\, dr + \sigma\left(B_s^{\theta} - B_t^{\theta}\right)\right) \geq \exp\left(-\int_t^s \sigma\theta_r^*\, dr + \sigma\left(B_s^{\theta} - B_t^{\theta}\right)\right).$$

Thus, the monotonicity of conditional expectation (Billingsley, 1986, p.468, Theorem 34.2(iii)) implies that

$$\begin{aligned} & E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r\, dr + \sigma\left(B_s^{\theta} - B_t^{\theta}\right)\right) \Big| \mathscr{F}_t\right] \\ \geq\ & E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\, dr + \sigma\left(B_s^{\theta} - B_t^{\theta}\right)\right) \Big| \mathscr{F}_t\right] \\ =\ & \exp\left(-\int_t^s \sigma\theta_r^*\, dr\right)\exp\left(\frac{1}{2}\sigma^2(s-t)\right) \\ =\ & E^{Q^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\, dr + \sigma\left(B_s^{\theta^*} - B_t^{\theta^*}\right)\right) \Big| \mathscr{F}_t\right], \end{aligned}$$

where we invoked the “degeneracy” of (θ_t^*) to show the equalities. Consequently, (54) holds.

We can now rewrite W as follows.

$$\begin{aligned} & W(\pi_t, t) \\ =\ & \inf_{\theta\in\Theta} E^{Q^{\theta}}\left[\int_t^T e^{-\rho(s-t)}\pi_s\, ds \Big| \mathscr{F}_t\right] \\ =\ & \inf_{\theta\in\Theta} \int_t^T E^{Q^{\theta}}\left[e^{-\rho(s-t)}\pi_s \Big| \mathscr{F}_t\right] ds \\ =\ & \inf_{\theta\in\Theta} \int_t^T \pi_t \exp\left(\left(\mu - \rho - \frac{1}{2}\sigma^2\right)(s-t)\right) E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r\, dr + \sigma\left(B_s^{\theta} - B_t^{\theta}\right)\right) \Big| \mathscr{F}_t\right] ds \\ =\ & \int_t^T \pi_t \exp\left(\left(\mu - \rho - \frac{1}{2}\sigma^2\right)(s-t)\right) E^{Q^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\, dr + \sigma\left(B_s^{\theta^*} - B_t^{\theta^*}\right)\right) \Big| \mathscr{F}_t\right] ds \end{aligned} \tag{55}$$

$$
\begin{aligned}
&= \int_t^T \pi_t \exp\left(\left(\mu-\rho-\frac{1}{2}\sigma^2\right)(s-t)-\int_t^s \sigma\theta_r^*\,dr\right) E^{Q^{\theta^*}}\left[\exp\sigma\left(B_s^{\theta^*}-B_t^{\theta^*}\right)\middle|\mathscr{F}_t\right] ds \\
&= \int_t^T \pi_t \exp\left(\left(\mu-\rho-\frac{1}{2}\sigma^2\right)(s-t)-\int_t^s \sigma\theta_r^*\,dr\right) \exp\left(\frac{1}{2}\sigma^2(s-t)\right) ds \\
&= \int_t^T \pi_t \exp\left(-(\rho-\mu)(s-t)-\int_t^s \sigma\theta_r^*\,dr\right) ds,
\end{aligned}
$$

where the first equality holds by (18); the second equality holds by Fubini's theorem for conditional expectation (Ethier and Kurtz, 1986, p.74, Proposition 4.6)[23]; the third equality holds by (22); the forth equality is derived from (54); the fifth equality holds by the facts that (θ_t^*) is a "degenerate" stochastic process; and the sixth equality holds by the fact that $\left(B_t^{\theta^*}\right)$ is the Brownian motion with respect to Q^{θ^*} and the formula for the expectation of a lognormal distribution (see, for example, Mood, Graybill and Boes, 1974). Thus, Proposition 1 is proved. ■

A.2. Derivation of (30).

We obtain

$$
\begin{aligned}
& V_t \\
&= \max_{t'\geq t}\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I\,\middle|\,\mathscr{F}_t\right] \\
&= \max\left\{\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_t^T e^{-\rho(s-t)}\pi_s\,ds\,\middle|\,\mathscr{F}_t\right]-I,\ \max_{t'\geq t+dt}\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I\,\middle|\,\mathscr{F}_t\right]\right\} \\
&= \max\left\{W_t-I,\ \max_{t'\geq t+dt}\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I\,\middle|\,\mathscr{F}_t\right]\right\} \\
&= \max\left\{W_t-I,\ \max_{t'\geq t+dt}\min_{\theta\in\Theta} E^{Q^\theta}\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I\,\middle|\,\mathscr{F}_t\right]\right\}
\end{aligned}
$$

[23]Since the Brownian motion is a measurable process (Billingsley, 1986, p.530, Theorem 37.2) and (θ_t) is a measurable process by assumption, (π_t) is also a measurable process by (22), and hence, a function $(s,\omega)\mapsto e^{-\rho(s-t)}\pi_s(\omega)$ is $(\mathscr{B}([t,T])\otimes\mathscr{F}_T)$-measurable. Furthermore, it also holds from (22) that

$$\int_t^T E^{Q^\theta}[\,|e^{-\rho(s-t)}\pi_s|\,]\,ds < +\infty.$$

Therefore, we may invoke Ethier and Kurtz (1986, p.74, Proposition 4.6) to conclude that there exists a function $f:[t,T]\times\Omega\to\mathbb{R}$ such that f is $(\mathscr{B}([t,T])\otimes\mathscr{F}_t)$-measurable, $(\forall s,\omega)\ f(s,\omega)=E^{Q^\theta}[e^{-\rho(s-t)}\pi_s|\mathscr{F}_t](\omega)$, $\int_t^T|f(s,\omega)|\,ds < +\infty$ P-a.s., and

$$(\forall\omega)\quad \int_t^T f(s,\omega)\,ds = E^{Q^\theta}\left[\int_t^T e^{-\rho(s-t)}\pi_s\,ds\,\middle|\,\mathscr{F}_t\right](\omega),$$

justifying the second equality in (55).

$$
\begin{aligned}
&= \max\left\{W_t - I, e^{-\rho dt} \max_{t' \geq t+dt} \min_{\theta\in\Theta} E^{Q^\theta}\left[E^{Q^\theta}\left[\int_{t'}^{T} e^{-\rho(s-t-dt)}\pi_s\, ds - e^{-\rho(t'-t-dt)} I \,\middle|\, \mathcal{F}_{t+dt}\right] \middle|\, \mathcal{F}_t\right]\right\} \\
&= \max\left\{W_t - I, e^{-\rho dt} \max_{t' \geq t+dt} \min_{\theta\in\Theta} E^{Q^\theta}\left[\min_{\theta'\in\Theta} E^{Q^{\theta'}}\left[\int_{t'}^{T} e^{-\rho(s-t-dt)}\pi_s\, ds - e^{-\rho(t'-t-dt)} I \,\middle|\, \mathcal{F}_{t+dt}\right] \middle|\, \mathcal{F}_t\right]\right\} \\
&= \max\left\{W_t - I, e^{-\rho dt} \min_{\theta\in\Theta} E^{Q^\theta}\left[\max_{t' \geq t+dt} \min_{\theta'\in\Theta} E^{Q^{\theta'}}\left[\int_{t'}^{T} e^{-\rho(s-t-dt)}\pi_s\, ds - e^{-\rho(t'-t-dt)} I \,\middle|\, \mathcal{F}_{t+dt}\right] \middle|\, \mathcal{F}_t\right]\right\} \\
&= \max\left\{W_t - I, e^{-\rho dt} \min_{\theta\in\Theta} E^{Q^\theta}\left[V_{t+dt} \mid \mathcal{F}_t\right]\right\} \\
&= \max\left\{W_t - I, (1-\rho\, dt)\left(\min_{\theta\in\Theta} E^{Q^\theta}\left[dV_t \mid \mathcal{F}_t\right] + V_t\right)\right\} \\
&= \max\left\{W_t - I, \min_{\theta\in\Theta} E^{Q^\theta}\left[dV_t \mid \mathcal{F}_t\right] + V_t - \rho V_t dt\right\},
\end{aligned}
$$

where each equality holds by: the definition of V_t, (29) (first); splitting the decision between to invest now (that is, at time t) and to wait for a short time interval and reconsider whether to invest or not after it (that is, at time $t+dt$) (second); the definition of W_t, (26) (third); the definition of $\mathcal{P}^\Theta$, (18) (fourth); the law of iterated integrals (fifth); the rectangularity, (24) (sixth); the fact that t' is restricted to be greater than or equal to $t+dt$ (seventh); the definition of V_t, (29), with t replaced by $t+dt$ (eighth); writing V_{t+dt} as $V_t + dV_t$ and approximating $e^{-\rho dt}$ by $(1-\rho dt)$ (such an approximation is justified since we let dt go to zero) (ninth); and eliminating the term which is of a higher order than dt (tenth). ■

A.3. Derivation of (40).

We get

$$
\begin{aligned}
&\min_{Q\in\mathcal{P}^\Theta} E^Q\left[dV_t \mid \mathcal{F}_t\right] = \min_{\theta\in\Theta^K} E^{Q^\theta}\left[dV_t \mid \mathcal{F}_t\right] \\
&= \min_{\theta\in\Theta^K} E^{Q^\theta}\left[V'(W_t)\left((\mu-\sigma\theta_t)W_t dt + \sigma W_t dB_t^\theta\right) + \frac{1}{2}\sigma^2 W_t^2 V''(W_t)\, dt \,\middle|\, \mathcal{F}_t\right] \\
&= \min_{\theta\in\Theta^K} V'(W_t)(\mu-\sigma\theta_t)W_t dt + \frac{1}{2}\sigma^2 W_t^2 V''(W_t)\, dt \\
&= V'(W_t)(\mu-\sigma\theta^*)W_t dt + \frac{1}{2}\sigma^2 W_t^2 V''(W_t)\, dt,
\end{aligned}
$$

where the first equality holds by the definition of $\mathcal{P}^\Theta$; the second equality holds by the fact that (39) holds for each θ; the third equality holds by the fact that $\left(B_t^\theta\right)$ is the Brownian motion with respect to Q^{θ}; and the last equality holds by the definition of θ^* (that is, (31)) and the conjecture that V' is positive. ■

A.4. Derivation of (45), (46) and (47) as a solution of (41).

To solve (41) with (42), (43) and (44), consider the following quadratic equation called the *characteristic equation* for (41):

$$\frac{1}{2}\sigma^2 x(x-1) + (\mu - \sigma\theta^*)x - \rho = 0,$$

the solutions to which are given by α defined by (47) and its conjugate, β. It turns out that $\alpha > 1$ and $\beta < 0$ (see A.5 below). Furthermore, it can be easily verified that both W_t^{α} and W_t^{β} solve (41) and that the Wronskian of W_t^{α} and W_t^{β} is nonzero for any $W_t > 0$. (Here, the *Wronskian* of two functions f_1 and f_2 is defined by $W(f_1, f_2) = f_1 f_2' - f_1' f_2$, and $W(W_t^{\alpha}, W_t^{\beta}) = (\beta - \alpha)W_t^{\alpha+\beta-1}$.) Hence, any solution to (41) can be expressed uniquely as a linear combination of W_t^{α} and W_t^{β}, that is,

$$V(W_t) = AW_t^{\alpha} + BW_t^{\beta}, \tag{56}$$

where A and B are some reals (Boyce and DiPrima, 1986, p.116, Theorem 3.4). Conversely, it is obvious that any function of the form (56) is a solution to (41). We conclude that (56) exhausts all the solutions to (41).

We now turn to the boundary conditions. The negativity of β and (42) immediately imply that $B = 0$ and hence

$$V(W_t) = AW_t^{\alpha}, \tag{57}$$

where A still remains undetermined.

Next, (57), (43) and (44) imply the following two equations:

$$A(W^*)^{\alpha} = W^* - I \quad \text{and} \quad \alpha A(W^*)^{\alpha-1} = 1.$$

By solving these equations, we obtain the solution given in the text. ■

A.5. Proof of $\alpha < 1$ and $\beta < 0$.

To see that $\alpha < 1$, note that

$$\begin{aligned}\alpha &> \frac{-\left(\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right)^2+2(\mu-\sigma\theta^*)\sigma^2}}{\sigma^2} \\ &= \frac{-\left(\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right)+\left|\mu-\sigma\theta^*+\frac{1}{2}\sigma^2\right|}{\sigma^2}\end{aligned}$$

where the strict inequality holds since $\rho > \mu - \sigma\theta^*$ by the assumption that $\rho > \mu$ and $0 \in K$. The claim follows since the last term is unity if $\mu - \sigma\theta^* \geq -\frac{1}{2}\sigma^2$ and it is more than unity if $\mu - \sigma\theta^* < -\frac{1}{2}\sigma^2$.

Also, β, the conjugate of α, is negative since

$$\begin{aligned}\beta &= \frac{-\left(\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right)-\sqrt{\left(\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right)^2+2\rho\sigma^2}}{\sigma^2} \\ &< \frac{-\left(\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right)-\left|\mu-\sigma\theta^*-\frac{1}{2}\sigma^2\right|}{\sigma^2} \leq 0\end{aligned}$$

where the strict inequality holds since $\rho > 0$. ■

Appendix B: Recursive Structure under Strong Rectangularity

In this appendix, we prove that recursive structure (24) holds true under strong rectangularity.

Let θ be a density generator, let $\left(z_t^\theta\right)$ be defined by (15) and define the measure Q_t^θ by

$$(\forall t \in [0,T])(\forall A \in \mathscr{F}_T) \quad Q_t^\theta(A) = \int_A z_t^\theta \, dP\,.$$

Then, the next lemma shows that Q_t^θ is a probability measure which coincides with Q^θ over $\mathscr{F}_t$. This is a direct implication of the martingale property of $\left(z_t^\theta\right)$.

Lemma B 1. *The measure Q_t^θ is a probability measure satisfying*

$$(\forall A \in \mathscr{F}_t) \quad Q_t^\theta(A) = Q^\theta(A)\,,$$

where Q^θ is defined by (17).

Proof. Since $\left(z_t^\theta\right)$ is a martingale, it follows that $Q_t^\theta(\Omega) = E^P\left[z_t^\theta\right] = z_0^\theta = 1$, and hence, Q_t^θ is certainly a probability measure. To show the claimed equality, let $t \in [0,T]$ and let $A \in \mathscr{F}_t$. Then,

$$Q_t^\theta(A) = \int_A z_t^\theta \, dP = \int_A E^P\left[z_T^\theta \middle| \mathscr{F}_t\right] dP = \int_A z_T^\theta \, dP = Q^\theta(A),$$

where the second equality holds by the fact that $\left(z_t^\theta\right)$ is a martingale and the third equality holds by the definition of the conditional expectation and the assumption that $A \in \mathscr{F}_t$. ■

Lemma 1 implies the next lemma showing that, loosely speaking, the expectation with respect to Q^θ of an $\mathscr{F}_t$-measurable random variable conditional on $\mathscr{F}_s$ depends on the density generator θ only between s and t.

Lemma B 2. *Let $0 \le s \le t \le T$ and let x be a $\mathscr{F}_t$-measurable random variable. Then, $E^{Q^\theta}[x \mid \mathscr{F}_s]$ depends only on $(\theta_u)_{s \le u < t}$.*

Proof. First, note that

$$E^{Q^\theta}[x \mid \mathscr{F}_s] = \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathscr{F}_s\right]. \tag{58}$$

To see this, let $A \in \mathscr{F}_s$. Then,

$$\begin{aligned}
\int_A \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathscr{F}_s\right] dQ^\theta &= \int_A \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathscr{F}_s\right] dQ_s^\theta \\
&= \int_A \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathscr{F}_s\right] z_s^\theta \, dP \\
&= \int_A E^P\left[x z_t^\theta \middle| \mathscr{F}_s\right] dP \\
&= \int_A x z_t^\theta \, dP \\
&= \int_A x \, dQ_t^\theta \\
&= \int_A x \, dQ^\theta,
\end{aligned}$$

where each equality holds by: Lemma B1 and the fact that the integrand is $\mathscr{F}_s$-measurable (first); the definition of Q_s^θ (second); a cancellation (third); the definition of conditional expectation and the fact that $A \in \mathscr{F}_s$ (fourth); the definition of Q_t^θ (fifth); Lemma B1 and the fact that the integrand is $\mathscr{F}_t$-measurable (sixth). Since the whole equality holds for any $A \in \mathscr{F}_s$, (58) follows by the definition of conditional expectation.

Second, the right-hand side of (58) is rewritten as

$$\frac{1}{z_s^\theta} E^P \left[x z_t^\theta \,\middle|\, \mathscr{F}_s \right] = E^P \left[x \exp\left(-\frac{1}{2} \int_s^t \theta_u \, du - \int_s^t \theta_u \, dB_u \right) \,\middle|\, \mathscr{F}_s \right]$$

by the definition of $\left(z_t^\theta\right)$. This shows that $E^{Q^\theta}[x \mid \mathscr{F}_s]$ depends only on $(\theta_u)_{s \le u \le t}$. In fact, it depends only on $(\theta_u)_{s \le u < t}$ since stochastic processes $\left(\int_0^t \theta_u \, du\right)_t$ and $\left(\int_0^t \theta_u \, dB_u\right)_t$ have a continuous sample path for each $\omega \in \Omega$ (Karatzas and Shreve, 1991, p.140, Remark 2.11). This completes the proof. ■

Using the above two lemmas we prove the main result in this appendix.

Lemma B 3. *Let $0 \le s \le t \le T$ and let x be an $\mathscr{F}_T$-measurable random variable. Also, assume that Θ is strongly rectangular. Then, under the assumption that the minima exist, it holds that*

$$\min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right].$$

Proof. To show that "$\ge$" holds, let θ^* be such that

$$E^{Q^{\theta^*}} \left[E^{Q^{\theta^*}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right].$$

Then,

$$\begin{aligned}
\min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] &= E^{Q^{\theta^*}} \left[E^{Q^{\theta^*}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] \\
&\ge E^{Q^{\theta^*}} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] \\
&\ge \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right].
\end{aligned}$$

To show that "$\le$" holds, let θ^* and θ'^* be such that

$$E^{Q^{\theta^*}} \left[E^{Q^{\theta'^*}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right].$$

Also, define θ^{**} by: $(\theta_u^{**})_{0 \le u < t} = (\theta_u^*)_{0 \le u < t}$ and $(\theta_u^{**})_{t \le u \le T} = \left(\theta_u'^*\right)_{t \le u \le T}$. Then, strong rectangularity implies that $\theta^{**} \in \Theta$. Furthermore,

$$\min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] \quad = \quad E^{Q^{\theta^*}} \left[E^{Q^{\theta'^*}} [x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right]$$

$$
\begin{aligned}
&= E^{Q^{\theta^{**}}}\left[E^{Q^{\theta^{**}}}[x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right] \\
&\geq \min_{\theta\in\Theta} E^{Q^{\theta}}\left[E^{Q^{\theta}}[x \mid \mathscr{F}_t] \,\middle|\, \mathscr{F}_s \right],
\end{aligned}
$$

where the second equality holds by Lemma B2 and the inequality holds by the fact that $\theta^{**} \in \Theta$. ■

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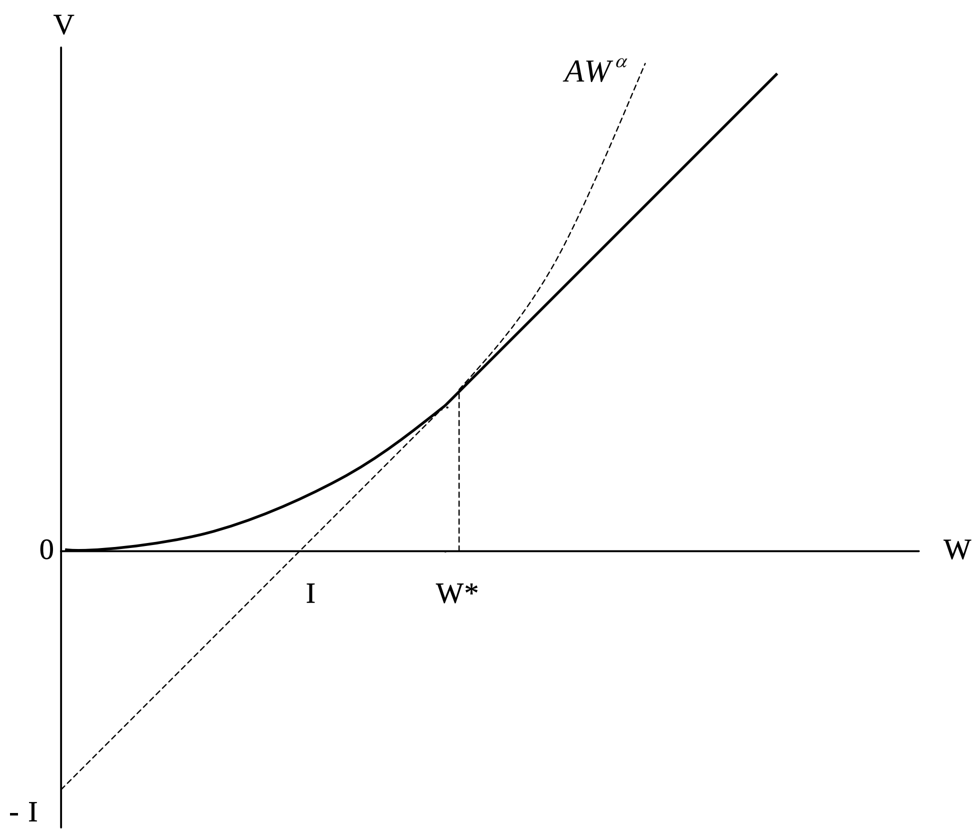

FIGURE 1 : Value Function

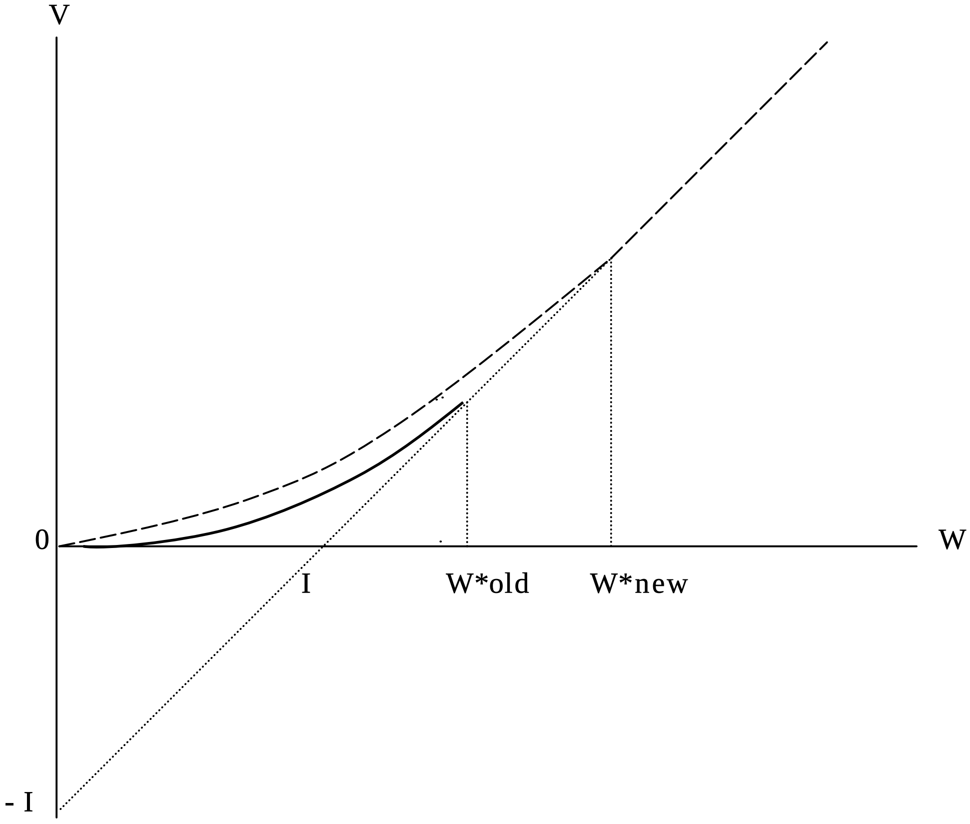

FIGURE 2 : An Increase in Risk

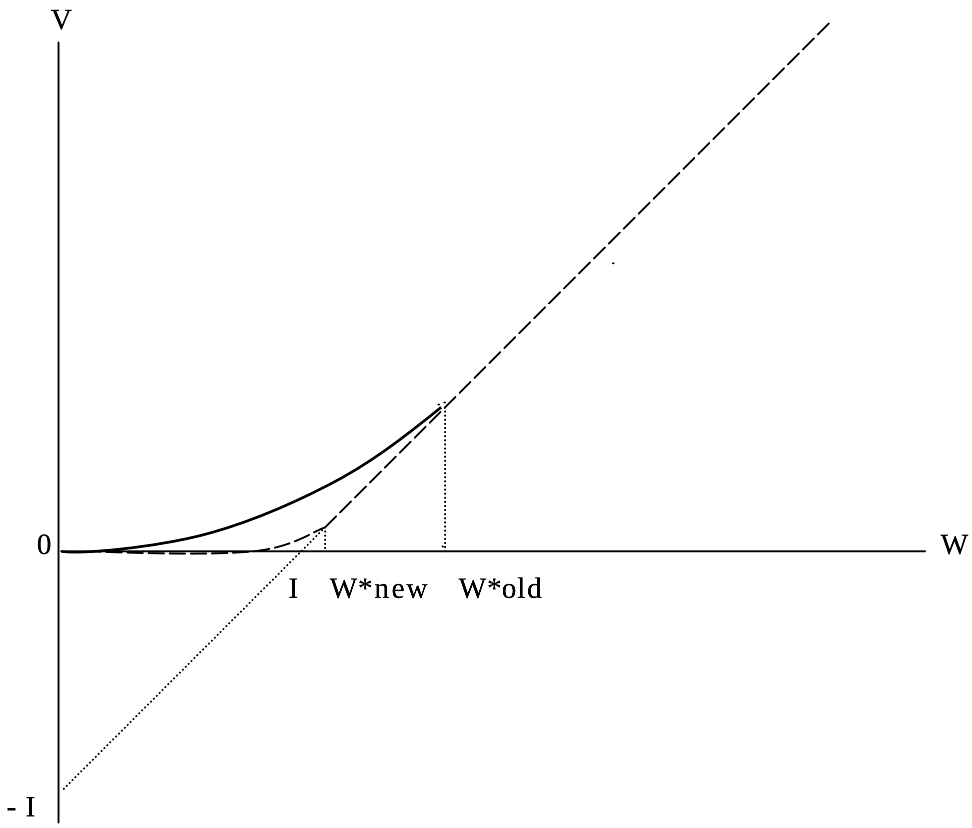

FIGURE 3 : An Increase in Knightian Uncertainty

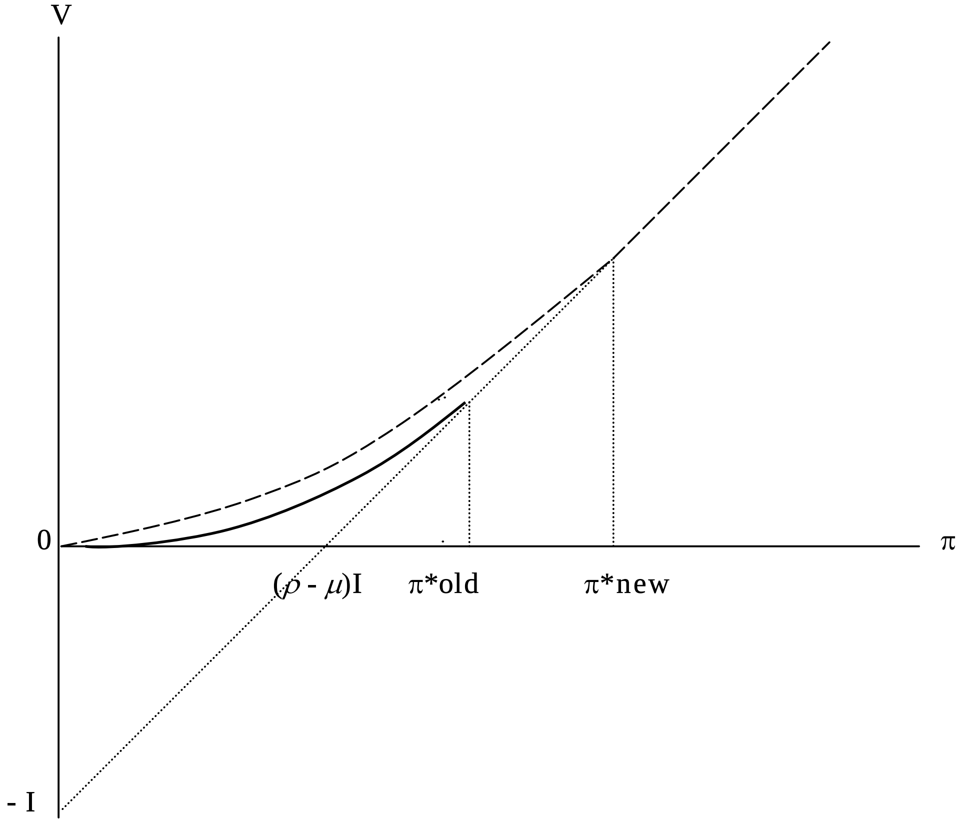

FIGURE 4 : An Increase in Risk
(x-axis is measured in terms of profit flow)

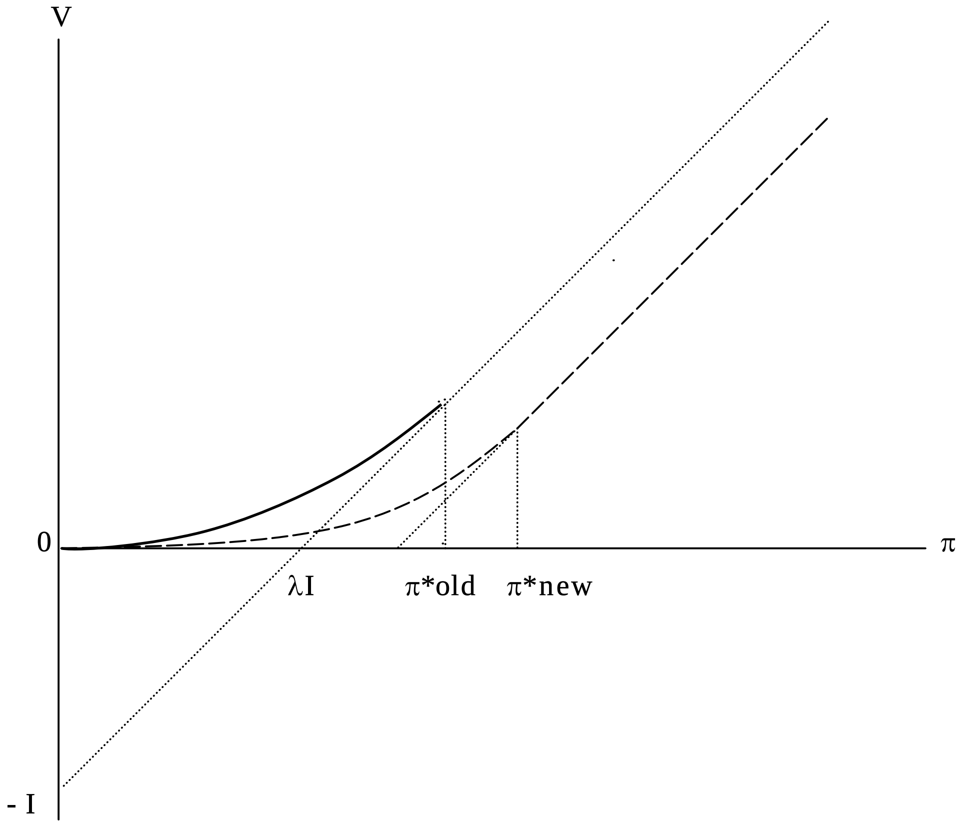

FIGURE 5 : An Increase in Knightian Uncertainty
(x-axis is measured in terms of profit flow)