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On RegARIMA Model, RegSSARMA Model and Seasonality

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Abstract

In the recent X-12-ARIMA program developed by the United States Census Bureau for seasonal adjustments, the RegARIMA modeling has been extensively utilized. We shall discuss some problems in the RegARIMA modeling when the time series are realizations of non-stationary integrated stochastic processes. We propose to use the seasonal switching autoregressive moving average (SSARMA) model and the regression SSARMA (RegSSARMA) model to cope with seasonalities commonly observed in many economic time series.

Key Words

X-12-ARIMA, Seasonal Adjustments, RegARIMA Models, Spurious Seasonal Non-stationarity, Seasonal SARMA model, RegSSARMA Model.

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1. Introduction

The analysis of seasonality in major economic time series has been one of important applications of modern statistical time series analysis. Since the central statistical offices in governments including the United States as well as Japan publish monthly and quarterly official statistics, this problem has been not only important in scientific point of view, but also in practical applications for policy evaluations and decisions. Most statistics offices in advanced countries in the world need some device to handle economic seasonality in order to make the seasonal adjustments before they publish final seasonally adjusted data. In this respect, the most important development in recent years is the new release of the X-12-ARIMA program by the research group of the Census Bureau in the United States Government. In this program the statistical time series model called the RegARIMA model has been extensively utilized. See Findley et. al. (1998) for its details.

The first purpose of this paper is to investigate important issues on the RegARIMA modeling. When we fit the RegARIMA model to actual economic time series, the estimated time series are often regarded as realizations of seasonally integrated time series. Then there is a natural question whether these actual seasonality should be treated as realizations of seasonal non-stationary time series. If the economic time series contained integrated seasonalites, then the variance of seasonal component should become large as time goes on and it should be roughly proportional to the data horizon. In many seasonal economic time series, however, it seems that the seasonal patterns fluctuate over time but do not change very wildly as the random walk models or the integrated stochastic processes predict. Although the fitting the seasonal random walk and the integrated stochastic models are often satisfactory, they could be spurious in some sense and there can be some other ways to describe seasonality in actual economic time series. If it is the case, there are important implications on statistical modeling including the prediction of seasonal economic time series and the seasonal adjustment procedures in particular.

In this paper we shall propose to use the seasonal switching ARMA (SSARMA) models and the regression SSARMA (RegSSARMA) models to handle seasonality in economic time series. They are simple extensions of the univariate seasonal ARMA models and the seasonal RegARMA models, which can have stationary seasonal components with the non-stationary trend components. We shall argue that the SSAR models are quite flexible to include the possible seasonal time series models as special cases. Also we shall illustrate why we often find the seasonal unit roots components when we fit the standard seasonal ARIMA models by using some theoretical results as well as real empirical examples. There will be some applications of the SSAR models and the RegSSAR models to deal with actual economic seasonality including the statsitical seasonal adjustment procedures in practice.

In Section 2 we shall analyze one important problem of estimating the regression part in the RegARIMA models and point out the resulting difficulty. Then in Section 3 we shall introduce the SSARMA and RegSSARMA models and discuss their statistical properties. In Section 4 we shall report some empirical results on economic time series including the macro consumption data in Japan and the well-known airline traffic data of Box and Jenkins (1971). Finally some concluding remarks will be given in Section 5. The proof of some theoretical results will be given in the Appendix.

2. RegARIMA model and Non-stationarity

We consider the class of the RegARIMA model developed by the U.S. Census Bureau and explained by Findley et. al. (1998). Let $\{y_t, t = 0, 1, \cdots\}$ be the univariate time series satisfying

$$(2.1) \qquad \phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D[y_t - \sum_{j=1}^{r}\beta_j z_{jt}] = \theta_q(B)\Theta_Q(B^s)\sigma v_t \ ,$$

where B is the backward shift operator ($By_t = y_{t-1}$), p, q, d, s, P, D, Q are non-negative integers ($s \geq 2$), and $\{z_{jt}, j = 1, \cdots, r\}$ are the set of explanatory variables. The associated lag polynomials of the seasonal ARIMA model are given by

$$(2.2) \qquad \begin{aligned} \phi_p(B) &= 1 - \phi_1 B - \cdots - \phi_p B^p \ , \\ \Phi_P(B^s) &= 1 - \Phi_1 B^s - \cdots - \Phi_P B^{sP} \ , \\ \theta_q(B) &= 1 - \theta_1 B - \cdots - \theta_q B^q \ , \\ \Theta_Q(B^s) &= 1 - \Theta_1 B^s - \cdots - \Theta_Q B^{sQ} \ , \end{aligned}$$

where $\{\phi_j\}$, $\{\Phi_j\}$ $\{\theta_j\}$, $\{\Theta_j\}$ are the unknown coefficient parameters of the ARIMA part, σ (> 0) is the standard deviation parameter, and $\{\beta_j\}$ are the unknown coefficient parameters of the regression part.
We assume that
(i) the absolute values of solutions of the corresponding equations

$$(2.3) \qquad \phi_p(z) = 0 \ , \Phi_P(z) = 0 \ , \theta_q(z) = 0 \ , \Theta_Q(z) = 0$$

are greater than 1, and
(ii) $\{v_t\}$ are a sequence of independently and identically distributed random variables with $E(v_t) = 0, E(v_t^2) = 1$, and the density function is positive almost everywhere in $\mathbf{R}$.

The stochastic processes defined by the RegARIMA model can be non-stationary and there are non-trivial issues on the statistical estimation of unknown parameters of the regression part in particular. We first illustrate the problem of our concern by using a simple example.

Example 1

In the X-12-ARIMA procedure there are some procedures such as the level shift and the detection of change points. For the simplicity, let $\{y_t, t = 0, 1, \cdots\}$ be the univariate time series satisfying

$$(2.4) \qquad (1-B^s)[y_t - \beta_0 - \beta_1 z_t] = \sigma v_t \ ,$$

where the simple explanatory variable $\{z_t\}$ is defined by $z_t = -1$ ($0 \leq t < [\lambda T]$), $z_t = 0$ ($[\lambda T] \leq t \leq T$), and $[\lambda T]$ ($0 < \lambda < 1$) is the period of level shift. By following the explanation of the RegARIMA model in Findley et. al. (1998), this model should be written as

$$(2.5) \qquad (1-B^s)y_t = \beta_1(1-B^s)z_t + \sigma v_t \ .$$

Then Findley et. al. (1998) recommend to use the standard regression technique with stationary disturbances to estimate the unknown parameter β_1 in this simple RegARIMA model.
By ignoring the initial conditions, in this case we have the explanatory variable defined by

$$z_t^* = (1-B^s)z_t = \begin{cases} 0 & \text{if } 0 \le t < [\lambda T] \\ 1 & \text{if } t = [\lambda T], \cdots, [\lambda T]+s-1 \\ 0 & \text{if } [\lambda T]+s < t \le T \end{cases} . \tag{2.6}$$

Then we immediately observe that the standard regression procedure including the t-statistic is only valid when the sequence of disturbance terms follow the normal distribution in the strict sense. If it was not satisfied, it seems that there are no strong reasons why we should use the standard regression procedures including statistical testing. Also there is no large sample justification for this procedure because the standard conditions on the regression variables cannot be satisfied. (See Section 2.6 of Anderson (1971) on the standard conditions for asymptotic theory, for instance.)

More generally, we consider the estimation problem of regression coefficients when the disturbance terms follow the Seasonal ARIMA model as

$$y_t = \beta^{'}\mathbf{z}_t + u_t \ , \tag{2.7}$$

where $\beta^{'}\mathbf{z}_t = \sum_{i=1}^{r}\beta_i z_{it}$ and $\{\mathbf{z}_t = (z_{it})\}$ are the vector of fixed regressors. In the purely formal way we could write

$$u_t = [\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D]^{-1}[\theta_q(B)\Theta_Q(B^s)]\sigma v_t \ . \tag{2.8}$$

In the standard large sample theory we usually assume the conditions that
(iii) the vector sequence of explanatory variables $\mathbf{z}_t = (z_{it})$ satisfy

$$\frac{1}{n}\sum_{j=1}^{n}\mathbf{z}_{(j-1)s+i}\mathbf{z}_{(j-1)s+i}^{'} \to \mathbf{M}_i = \int_0^1 \mathbf{z}_i(t)\mathbf{z}_i^{'}(t)dt \tag{2.9}$$

and

$$\frac{1}{n}\max_{1\le j\le n, 1\le i\le s}\|\mathbf{z}_{(j-1)s+i}\|^2 \to 0 \tag{2.10}$$

as $n \to \infty$, where $sn = T$, $\mathbf{z}_i(t)$ is the $r\times 1$ vector depending on t $(0 \le t \le 1, i = 1, \cdots, s)$ and $\mathbf{M} = \sum_{i=1}^{s}\mathbf{M}_i$ is a positive definite matrix.
Let the t-statistic for the k-th regression coefficient β_k $(k = 1, \cdots, r)$ be defined by

$$t(\beta_k) = \frac{\hat{\beta}_k - \beta_k}{\sqrt{\hat{\sigma}_{LS}^2\mathbf{e}_k^{'}(\sum_{t=1}^{T}\mathbf{z}_t\mathbf{z}_t^{'})^{-1}\mathbf{e}_k}} \ , \tag{2.11}$$

where $\hat{\beta}_k$ is the least squares (LS) estimator of β_k , $\hat{\sigma}_{LS}^2$ is the standard LS estimator of σ^2 , and $\mathbf{e}_k = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the unit vector with 1 at the k-th component. Then we have the weak convergence result on the t-statistic and the sketch of its proof is given in the Appendix.

Theorem 2.1 *In the RegARIMA given by (2.1) with $d = 0, D \geq 1$, we assume (i), (ii) and (iii). Then the limiting random variable of $\frac{1}{\sqrt{T}} t(\beta_k)$ as $T \to +\infty$ can be written as*

$$t_k^* = \frac{\mathbf{e}_k' \mathbf{M}^{-1} \int_0^1 [\sum_{i=1}^s \mathbf{z}_i(r) \bar{B}_i(r)] dr}{\sqrt{\mathbf{e}_k' \mathbf{M}^{-1} \mathbf{e}_k} \sqrt{s \int_0^1 \sum_{i=1}^s \bar{B}_i^2(r) dr - \int_0^1 [\sum_{i=1}^s \mathbf{z}_i(r) \bar{B}_i(r)]' dr \mathbf{M}^{-1} \int_0^1 [\sum_{i=1}^s \mathbf{z}_i(r) \bar{B}_i(r)] dr}} , \tag{2.12}$$

where $\bar{B}_i(r_D)$ $(i = 1, \cdots, s)$ have the Ito's multiple integral representation as

$$\bar{B}_i(r_0) = \int_0^{r_0} \cdots \int_0^{r_{D-1}} dB_i(r_D) \prod_{l=1}^{D-1} dr_{D-l} \tag{2.13}$$

and $B_i(t)$ are the independent Standard Brownian Motions on $[0, 1]$.

For the definition and properties of the Brownian Motions and Ito's multiple integral representations, see Ikeda and Watanabe (1989) as the standard reference in stochastic analysis. In the above theorem we have treated the case when $d = 0$. It is also straightforward to extend the above representation theorem to the case when $d \geq 1$.

Corollary 2.1 *In the RegARIMA given by (2.1) with $d \geq 1, D \geq 1$, we assume (i), (ii) and (iii). Then the limiting random variable of $\frac{1}{\sqrt{T}} t(\beta_k)$ as $T \to +\infty$ can be written as*

$$t_k^* = \frac{\mathbf{e}_k' \mathbf{M}^{*-1} \int_0^1 [\mathbf{z}^*(r) \bar{B}^*(r)] dr}{\sqrt{\mathbf{e}_k' \mathbf{M}^{*-1} \mathbf{e}_k} \sqrt{\int_0^1 \bar{B}^{*2}(r) dr - \int_0^1 [\mathbf{z}^*(r) \bar{B}^*(r)]' dr \ \mathbf{M}^{*-1} \ \int_0^1 [\mathbf{z}^*(r) \bar{B}^*(r)] dr}} , \tag{2.14}$$

where $\mathbf{M}^* = (1/s)\mathbf{M}$, $\mathbf{z}^*(r) = (1/s) \sum_{i=1}^s \mathbf{z}(r)$, and $\bar{B}^*(r)$ $(0 \leq r \leq 1)$ have the Ito's multiple integral representation as

$$\bar{B}^*(r_0) = \int_0^{r_0} \cdots \int_0^{r_{d-1}} \sum_{i=1}^s \int_0^{r_d} \cdots \int_0^{r_{d+D-1}} dB_i(r_{d+D}) \prod_{l=1}^{d+D-1} dr_{D+d-l} \prod_{l'=1}^{d} dr_{d+1-l'} . \tag{2.15}$$

There are two distinctive features on the distribution of random variable t_k^* $(k = 1, \cdots, s)$. The first one is that we need the normalizing factor $1/\sqrt{T}$ for the underlying t-statistic. The second one is that the limiting distribution is significantly different from the standard t-distribution as well as the standard normal distribution. We have done a number of simulations to generate the limiting distribution function of t_k^* $(k = 1, \cdots, s)$ and investigate their statistical properties. We have found that they are skewed considerably depending on the integration orders d and D.

3. The SSARMA and RegSSARMA Models

In this section we shall introduce a new class of seasonal switching autoregressive moving-average (SSARMA) models and the regression SSARMA models. We denote the latter class as the RegSSARMA model which includes the class of the RegARIMA models as a special case. The main feature of the SSARMA models is a simple nonlinear time series model which includes the class of the seasonal ARIMA models as a special case and also they can handle the "spurious seasonal non-stationarity" in many cases.

3.1 Seasonal Switching ARMA model

Let $\{y_t, t = 0, 1, \cdots\}$ be the univariate time series satisfying

$$\phi_p(B)\sum_{i=1}^{s}\Phi_P^i(B^s)I_t^i[y_t - \sum_{j=1}^{r}\beta_j z_{jt}] = \theta_q(B)[\sum_{i=1}^{s}\Theta_Q^i(B^s)I_t^i\sigma_i]v_t \ , \tag{3.1}$$

where p, q, s, P, Q are non-negative integers ($s \geq 1$), and σ_i (> 0) are the unknown seasonal standard deviations, I_t^i are the seasonal indicator functions, and $\{z_{it}\}$ are the set of explanatory variables. The associated polynomials of the ARMA part are given by

$$\begin{aligned} \phi_p(B) &= 1 - \phi_1 B - \cdots - \phi_p B^p \ , \\ \Phi_P^i(B^s) &= 1 - \Phi_1^i B^s - \cdots - \Phi_P^i B^{sP} \ , \\ \theta_q(B) &= 1 - \theta_1 B - \cdots - \theta_q B^q \ , \\ \Theta_Q^i(B^s) &= 1 - \Theta_1^i B^s - \cdots - \Theta_Q^i B^{sQ} \ , \end{aligned} \tag{3.2}$$

where $\{\phi_j\}, \{\Phi_j^i\}$, $\{\theta_j\}, \{\Theta_j^i\}$ are the unknown coefficients, and $\{\beta_j\}$ are unknown regression coefficients.
We assume that
(i) the absolute values of solutions of the corresponding characteristic equations

$$\phi_p(z) = 0 \ , \Phi_P^i(z) = 0 \ , \theta_q(z) = 0 \ , \Theta_Q^i(z) = 0 \tag{3.3}$$

are greater than 1, and
(ii) $\{v_t\}$ are a sequence of independently and identically distributed random variables with $E(v_t) = 0$, $E(v_t^2) = 1$, and the density function is positive almost everywhere in $\mathbf{R}$.
In the above notation the indicator function I_t^i can be defined by $I_t^i = 1$ when t is in the i-th season of some year and $I_t^i = 0$ otherwise. In this formulation the SSARMA model and the RegARMA model are slightly different from the existing linear seasonal time series models. The stochastic processes in the discrete time defined by the SSARMA model can be stationary in many cases and we illustrate the distinctive features of the SSARMA model by using a simple example.

Example 2

Let $\{y_t, t = 0, 1, \cdots\}$ be the univariate time series satisfying

$$y_t = \sum_{i=1}^{s} I_t^i[a_i + b_i y_{t-s} + \sigma_i v_t] \ , \tag{3.4}$$

where $\{a_i\}, \{b_i\}, \{\sigma_i\}$ are unknown parameters and we denote this model simply as the SSAR(1) model.
When we have the stability conditions

$$\max_{i=1,\cdots,s} |b_i| < 1 \ , \tag{3.5}$$

then the SSAR(1) model can be re-written as

$$\sum_{i=1}^{s}(1-b_iB^s)I_t^i[y_t-\mu_i]=[\sum_{i=1}^{s}\sigma_iI_t^i]v_t\ , \tag{3.6}$$

where $(1-b_i)\mu_i=a_i\ (i=1,\cdots,s)$.

The most important feature of the SSARMA model is the mechanism of seasonal switchings and it can be natural to be regarded as a kind of non-linear phenomena. The unrestricted form of the SSAR(1) model, for instance, has $3s$ unknown parameters and it nests many existing linear seasonal time series models, i.e., the seasonal ARIMA models of Box and Jenkins (1971). If we have the equal seasonal variances, $\sigma_i=\sigma$, then the SSAR(1) model becomes the seasonal random coefficient models. If we further have the equal coefficients, i.e. $a_i=a$ and $b_i=b$, then we have the linear seasonal AR model in the Box-Jenkins approach. We have the standard non-stationary (seasonal) integrated process only if

$$b_i=1\ (i=1,\cdots,s) \tag{3.7}$$

and $a_i=a\ ,\sigma_i=\sigma$ at the same time.

3.2 Statistical Properties of the SSARMA model

Because the SSARMA model is a kind of non-linear time series model with switching mechanism, we need to investigate its statistical properties. For the SSAR(1) model explained in the last subsection for instance, we take the s-dimensional state space vector for Example 2 as $Y^{'}(t)=(y_t,y_{t-1},\cdots,y_{t-s+1})$ for $t=1,\cdots,T$. Then we have the Markovian representation

$$Y(t)=\mathbf{a}(t)+\mathbf{B}(t)Y(t-1)+V(t) \tag{3.8}$$

where $\boldsymbol{a}(t)$ is an $s\times 1$ vector and $\boldsymbol{B}(t)$ is an $s\times s$ coefficient matrix

$$\boldsymbol{a}(t)=\begin{pmatrix}\sum_{i=1}^{s}a_iI_t^i\\ 0\\ \vdots\\ 0\end{pmatrix}\ ,\ \boldsymbol{B}(t)=\begin{pmatrix}0 & \cdots & 0 & \sum_{i=1}^{s}b_iI_t^i\\ 1 & & & 0\\ & \ddots & & \\ \mathbf{0} & & 1 & 0\end{pmatrix}\ ,$$

and $V(t)$ is an $s\times 1$ disturbance vector

$$V(t)=\begin{pmatrix}\sum_{i=1}^{s}\sigma_iI_t^i\\ 0\\ \vdots\\ 0\end{pmatrix}v_t\ .$$

By using (3.8), the associated characteristic equation for the Markovian representation can be given by

$$|\lambda\mathbf{I}_s-\mathbf{B}(t)|=0\ , \tag{3.9}$$

where $\mathbf{I}_s$ is the $s \times s$ identity matrix. Then we immediately notice that the absolute values of all characteristic roots of (3.9) are less than 1 if we have the condition

$$\max_{i=1,\cdots.s} |b_i| < 1 \ . \tag{3.10}$$

Alternatively, we take the s-dimensional state vector for Example 2 $Y_j^{'} = (y_{(j-1)s+s}, y_{(j-1)s+s-1}, \cdots, y_{(j-1)s+1})$ for $j = 1, \cdots, n$ and we take $T = ns$ for the resulting simplicity. Then we have the alternative Markovian representation as

$$Y_j = \mathbf{a} + \mathbf{B}_1 Y_{j-1} + V_j \ , \tag{3.11}$$

where $\boldsymbol{a}$ is an $s \times 1$ vector and $\boldsymbol{B}_1$ is an $s \times s$ coefficient matrix such that

$$\boldsymbol{a} = \begin{pmatrix} a_s \\ a_{s-1} \\ \vdots \\ a_1 \end{pmatrix} , \ \boldsymbol{B}_1 = \begin{pmatrix} b_s & 0 & \cdots & 0 \\ 0 & b_{s-1} & 0 & 0 \\ & \ddots & & \\ 0 & & & b_1 \end{pmatrix} ,$$

and V_j is an $s \times 1$ disturbance vector

$$V_j = \begin{pmatrix} \sigma_s v_{(j-1)s+s} \\ \sigma_{s-1} v_{(j-1)s+s-1} \\ \vdots \\ \sigma_1 v_{(j-1)s+1} \end{pmatrix} .$$

The associated characteristic equation for the Markovian stochastic process (3.11) can be given by

$$|\lambda \mathbf{I}_s - \mathbf{B}_1| = 0 \ , \tag{3.12}$$

and the absolute values of all characteristic roots are less than 1 under the same condition in (3.10).
More generally, we can construct the (s+p)-dimensional state vector for (3.1) as $Y_j^{'} = (y^*_{(j-1)s+s}, y^*_{(j-1)s+s-1}, \cdots, y^*_{(j-1)s-(p-1)})$ for $j = 1, \cdots, n$, where $y^*_j = y_t - \sum_{j=s+1}^{r} \beta_j z_{jt}$, $z_{it} = I_{it}$ $(i = 1, \cdots, s)$ and $r \geq s$. Then we have the following result and the proof is given in the Appendix..

Theorem 3.2 *Let $\{y_t\}$ follow (3.1) and we assume (ii). Then there exists a stationary solution for the (s+p)-dimensional vector stochastic process $\{Y_j, j = 1, \cdots\}$ with $E[\|Y_j\|^2] < +\infty$ if and only if the absolute values of all solutions of*

$$\phi(z) = 0 \ , \ \Phi_P^i(z) = 0 \ (i = 1, \cdots, s) \tag{3.13}$$

are greater than one.

One important problem in the seasonal ARIMA (SARIMA) modeling has been the seasonal non-stationarity. When we fit the SARIMA models, we often find that the seasonal differencing looks appropriate in many economic time series. However, from the view of nonlinearity in the SSARMA modeling this phenomenon should be often interpreted as the "spurious seasonal integration" by using the univariate linear time series modeling.

Theorem 3.3 *Let $\{y_t\}$ follows the SSAR(1) model given by Example 2 with $E[v_t^4] < +\infty$. Assume the stability condition given by (3.10) and let*

$$\hat{b}_{LS} = \frac{\sum_{t=1}^{T}(y_{t-s} - \bar{y}_{-s})(y_t - \bar{y})}{\sum_{t=1}^{T}(y_{t-s} - \bar{y}_{-s})^2} , \tag{3.14}$$

where $T = sn$, $\bar{y} = (1/T)\sum_{t=1}^{T} y_t$, $\bar{y}_{-s} = (1/T)\sum_{t=1}^{T} y_{t-s}$ and the initial conditions y_t $(t \leq 0)$ are fixed.
(i) As $n \to \infty$ we have

$$\hat{b}_{LS} \xrightarrow{p} b^* = \frac{\sum_{i=1}^{s} \frac{b_i\sigma_i^2}{1-b_i^2} + \sum_{i=1}^{s}(\mu_i - \bar{\mu})^2}{\sum_{i=1}^{s} \frac{\sigma_i^2}{1-b_i^2} + \sum_{i=1}^{s}(\mu_i - \bar{\mu})^2} , \tag{3.15}$$

where $\bar{\mu} = (1/s)\sum_{i=1}^{s} \mu_i$.
(ii) If we have the sequences of $\mu_i = \mu_i(n), \sigma_i = \sigma_i(n)$ $(i = 1, \cdots, s)$ such that

$$\lambda(n, s) = \frac{\sum_{i=1}^{s}(\mu_i - \bar{\mu})^2}{\max_{i=1,\cdots,s}\{\sigma_i^2\}} \to \infty , \tag{3.16}$$

then we have $\hat{b}_{LS} \xrightarrow{p} 1$.

It is rather straightforward to extend this result to more general SSARMA models with some stability conditions. From our results we expect that we often find roots of the associated characteristic equations whose absolute values are near unity when we ignored some nonlinear seasonal factors in economic time series.

3.3 Estimation and Seasonal Modeling Procedure

There can be two alternative ways to estimate the SSARMA and the RegARMA models, that is, the least suares (LS) estimation method and the maximum likelihood (ML) estimation method.

If we use the SSAR models and the RegSSAR models, we have the direct Markovian representation for the state vector $\{Y_j, j = 1, \cdots, n\}$ $(T = sn)$ and use the standard LS estimation method for the regression model with stationary time series disturbance terms provided the conditions (i)-(iii) are satisfied. Then the classical asymptotic theory on the estimation of regression functions can be applicable. (See Section 2.6 of Anderson (1971).)

Alternatively, we can use the maximum likelihood (ML) estimation method for estimating the regression function and the seasonal unknown parameters. This method is more useful than the standard least squares method because we can use a series of statistical testing on the parameter restrictions in the RegSSARMA models and make the model selection procedure easier based on the Akaike's Information Criterion (AIC) developed by Akaike (1973). Because the class of the RegSSARMA models contain a large number of other seasonal time series models, we mainly have used the ML estimation in our data analysis.

3.4 Extensions of the SSARMA Models

In this paper we have introduced the class of regression seasonal switching ARMA models for analyzing economic time series analysis in practice. There can be other type of extensions of the SSAR models and we should briefly discuss such possibilities.

The traditional additive decomposition of seasonal time series can be represented as

$$y_t = T_t + S_t + I_t \ , \tag{3.17}$$

where T_t stands for the trend component, S_t stands for the seasonal component, and I_t stands for the irregular noise component. Kitagawa (1993), for instance, has proposed to use the random walk model for T_t and the seasonal random walk model for S_t in his DECOMP program. It is rather straightforward to use the stationary SSARMA models for S_t in his formulation. Then the SSAR modeling can be incorporated into the DECOMP program of seasonal adjustments which has been developed by using the state space representation for $\{y_t\}$.

4. Some Case Studies

In this section we shall report an empirical application using a set of macro economic time series data in Japan and the airline traffic data used by Box and Jenkins (1971) for the illustrative purposes.

In order to elliminate the deterministic trend and deterministic seasonal components, we use the regression part of the RegSSAR models as

$$T_t = \sum_{i=1}^{s} \beta_i D_{it} + \sum_{i=s+1}^{s+k} \beta_i t^{i-s} \ , \tag{4.1}$$

where D_{it} are the seasonal dummies, $\beta^{'} = (\beta_1, \cdots, \beta_{s+l})$ is the vector of regression coefficients, and $\mathbf{z}_t^{'} = (D_{1t}, \cdots, D_{st}, t, \cdots, t^k)$ are the vector of regressors [1] . We have used this formulation mainly because it is easy to treat the seasonal means and the likelihood calculations. The estimation of unknown parameters of the RegSSARMA models have been done by the maximum likelihood method and we restrict the highest order of time trend is 3 in order to avoid unstable trend estimation.

As the first empirical data analysis we use the quarterly time series data of macro consumption in Japan which has been an important macro variable published by the Cabinet office of Japan. The original data sets are the quarterly raw data from 2nd quarter of 1975 to 4th quarter of 2000. All data are transformed such that the level of the first data point is 100. By using the minimum AIC criterion, we have chosen $k = 3$. The estimated coefficients of the unrestricted RegSSAR model and their t-values are given in Table 1.

Also by using the RegARMA model under the assumption that $\mu_i = \mu$, $b_i = b$, and $\sigma_i = \sigma$ $(i = 1, \cdots, 4)$, we have the estimated result as

$$\begin{array}{rcccccl} y_t^* = & 105.3578 & + & 0.9348 y_{t-4}^* & + & 2.643 v_t & , \\ & (104.7) & & (34.2) & & & \end{array} \tag{4.2}$$

[1] When we use the time trend variables, the normalization factors for Conditions in (2.9) and (2.10) should be modified appropriately as Theorem 2.6.1 of Anderson (1971),

Table 1: Unrestricted SSARMA Model for Consumption

We use the maximum likelihood method under the assumption of stationary time series for estimating unknown parameters of the underlying models. The figures in the parentheses are the t-ratio's and the variances have been estimated by the second derivatives of the likelihood functions.

	Seasonal Dummy	AR(4)	σ_i^2
2nd Quarter	101.3410 (10.7137)	0.6246 (4.2474)	5.4642
3rd Quarter	106.6477 (20.3588)	0.6702 (4.7813)	3.4901
4th Quarter	118.8236 (94.5556)	0.6381 (4.2945)	5.5952
1st Quarter	101.1639 (315.8540)	0.4274 (2.3680)	8.0312
	AIC=500.53		

Table 2: A Restricted SSARMA Model for Consumption

	Seasonal Dummy	AR (4)	σ_i^2
2nd Quarter	101.4658 (134.92)	0.6438 (7.30)	6.3610
3rd Quarter	106.7785 (11.81)	0.6438 (7.30)	3.5028
4th Quarter	118.9281 (61.01)	0.6438 (7.30)	6.3610
1st Quarter	101.2872 (211.34)	0.4311 (2.70)	6.3610
	AIC=493.72		

and $AIC = 515.25$, where $y_t^* = y_t - \sum_{i=1}^{3} \beta_i t^i$ and the values in the parentheses are t-ratios.
We immediately observe that the estimated AR coefficient is quite near to the non-stationary region when we fit the RegARMA model. On the other hand, the estimated AR coefficients of the RegSSAR model are moderate and the estimated coefficients of the dummy variables are significantly different in each season. We also notice that the estimated coefficients of the first and 4th dummy variables are similar, and the estimated 4-th AR coefficient is different from other coefficients. Hence we could expect to have a more parsimonious RegSSARMA model than the unrestricted RegSSAR model. Then we have tried to impose some restrictions on coefficients and variances on the unrestricted RegSSAR model. After some trials and errors mainly by using the minimum AIC criterion, the best model in the class of the RegSSARMA model and its estimated result is given in Table 2.

As the second empirical data set, we have used famous airline traffic monthly data used by Box and Jenkins (1971). In this case we have chosen $k = 2$ by the minimum AIC criterion. The estimated coefficients of the unrestricted RegSSAR model and their t-values are given in Table 3.

Also by using the RegARMA model under the assumption that $\mu_i = \mu$, $b_i = b$,

Table 3: Unrestricted SSARMA Model for Airline Traffic Data

We use the maximum likelihood method under the assumption of stationary time series for estimating unknown parameters of the underlying models. The figures in the parentheses are the t-ratio's and the variances have been estimated by the second derivatives of the likelihood functions.

	Seasonal Dummy	AR (12)	σ_i^2
January	98.3693 (364.7219)	-0.2663 (-0.9540)	0.6188
February	98.0217 (145.9247)	0.4321 (1.6309)	1.7853
March	100.6701 (192.8839)	0.2696 (0.8097)	1.5956
April	100.0008 (307.8176)	-0.3071 (-1.0684)	1.2692
May	99.9298 (353.3433)	-0.4169 (-1.6142)	0.8681
June	102.5562 (244.7918)	0.4103 (1.5697)	0.6462
July	104.8678 (146.0224)	0.8104 (4.9099)	0.3569
August	104.5475 (177.1758)	0.7141 (3.6765)	0.4360
September	101.4833 (384.4140)	-0.0155 (-0.0699)	0.3441
October	98.5535 (377.2055)	-0.0681 (-0.2226)	0.3492
November	95.5138 (325.1808)	-0.0810 (-0.1129)	0.6012
December	97.9352 (321.3479)	-0.0363 (-0.0486)	0.6385
	AIC=429.43		

Table 4: A Restricted SSARMA Model for Airline Traffic Data

	Seasonal Dummy	AR (12)	σ_i^2
January	98.3620 (341.7981)	-0.2091 (-1.9113)	0.5664
February	98.0119 (119.2458)	0.5477 (4.8778)	1.5756
March	100.5922 (132.2600)	0.5477 (4.8778)	1.5756
April	99.9507 (265.3878)	-0.2091 (-1.9113)	1.5756
May	99.8754 (363.6894)	-0.2091 (-1.9113)	0.5664
June	102.5045 (109.6486)	0.5477 (4.8778)	0.5664
July	104.7370 (136.4556)	0.5477 (4.8778)	0.5664
August	104.4870 (93.4429)	0.5477 (4.8778)	0.5664
September	101.4150 (367.6975)	-0.2091 (-1.9113)	0.5664
October	98.4804 (363.7761)	-0.2091 (-1.9113)	0.5664
November	95.4472 (173.7319)	-0.2091 (-1.9113)	0.5664
December	97.8733 (345.8628)	-0.2091 (-1.9113)	0.5664
	AIC=399.66		

and $\sigma_i = \sigma$ $(i = 1, \cdots, 12)$, we have the estimated result as

$$y_t^* = \underset{(116.8150)}{100.8611} + \underset{(31.2683)}{0.8985 y_{t-12}^*} + 1.23 v_t \ , \tag{4.3}$$

and $AIC = 498.87$. We also observe that the estimated AR coefficient is quite near to the non-stationary region when we fit the RegARMA model. On the other hand, the estimated AR coefficients of the RegSSARMA model are moderate and the estimated coefficients of the dummy variables are significantly different from month to month. In particular, the July and August AR coefficients are relatively large and other coefficients are quite different from these two estimated coefficients which are not significantly different from zero. We also notice that the estimated coefficients of July and August dummy variables are similar, and they are larger than other estimated coefficients. Hence we could expect to have a more parsimonious RegSSARMA model than the unrestricted RegSSAR model. Then we have tried to impose some restrictions on coefficients and variances on the unrestricted RegSSAR model. After some trials and errors mainly by using the minimum AIC criterion, the best model in the class of the RegSSARMA model and its estimated result is given in Table 4.

5. Conclusions

In this paper we have discussed one important problem in the RegARIMA modeling which has been extensively used in the X-12-ARIMA seasonal adjustment program which has been developed by the U.S. Census Bureau. When we have non-stationary seasonal integrated processes, the estimation problem of the regression function in the RegARIMA models becomes non-standard. Then we have shown one weak convergence result on the asymptotic distribution of the t-statistic when the disturbance terms follow the seasonal ARIMA processes.

Then we have introduced a class of the seasonal switching ARMA (SSARMA) model and the RegSSARMA model in order to capture the non-linear seasonal patterns. We have argued that the SSARMA modeling is a simple way to handle the nonlinear seasonality as we have discussed. From our limited experiences we only need relatively simple SSARMA models to describe actual seasonality in many economic time series. In particular we need stationary SSARMA time series models to handle actual seasonality in many cases.

Finally, there are several problems remain to be investigated. It may be interesting to develop the SSARMA time series models with non-stationary stochastic trends. For instance, the non-parametric estimation problem of the trend functions in the non-linear seasonal time series analysis should be investigated.

6. Mathematical Appendix

In this Appendix, we gather some mathematical details and the proofs of Theorems which we have omitted in the previous sections.

Proof of Theorem 2.1 and Corollary 2.1 :
First we consider the case when

$$(1 - B)^d (1 - B^s)^D u_t = v_t \ \ (t = 1, 2, \cdots) \ , \tag{A.1}$$

where $d \geq 1, D \geq 0, \sigma = 1$, and $\{v_t\}$ are a sequence of i.i.d. random variables and the initial conditions are fixed as $u_{-s} = 0$ $(s \geq 0)$. We define a sequence of random variables

(A.2) $$u_t^{(k)} = u_{t-1}^{(k)} + u_t^{(k+1)} \ \ (k = 0, 1, \cdots, d-1),$$

where we denote $u_t^{(0)} = u_t, u_t^{(d+D)} = v_t$ and treat $s_{d-1}(T)$ as a function of T . Then we have the representation as

(A.3) $$u_t = \sum_{t \geq s_1 \geq \cdots \geq s_d \geq 1} u_{s_d}^{(d)}$$

and then we can decompose

(A.4) $$\sum_{s_d=1}^{s_{d-1}(T)} u_{s_d}^{(d)} = \sum_{i=1}^{s} \sum_{j=1}^{[\frac{s_{d-1}(T)}{s}]} u_{(j-1)s+i}^{(d)} + R(s_{d-1}(T)) ,$$

where we denote the remainder term as $R(s_{d-1}(T))$. We have the condition that $E[R(s_{d-1}(T))^2]$ is bounded uniformly with respect to T from our assumptions and we can ignore the initial conditions asymptotically in the following derivations. By using the substitution of seasonal decompositions for $u_{(j-1)s+i}^{(d)}$ $(j \geq 1, i = 1, \cdots, s)$, we can express

(A.5) $$u_t \sim u_t^* = \sum_{t \geq s_1 \geq \cdots \geq s_{d-1}} \sum_{i=1}^{s} \sum_{[\frac{s_{d-1}}{s}] \geq s_d \geq \cdots \geq s_{d+D} \geq 1} v_{(s_{d+D}-1)s+i} .$$

By using the weak convergence arguments and checking the tightness condition (which are quite similar to the proof of Theorem 3.1 of Tanaka (1996)), we have

(A.6) $$\frac{1}{T^{d+D-1}\sqrt{T}} = (\frac{1}{s})^{D-\frac{1}{2}} \frac{1}{T^d} \frac{1}{n^{D-\frac{1}{2}}} u_{t(T)}$$
$$\stackrel{w}{\Longrightarrow} (\frac{1}{s})^{D-\frac{1}{2}} \bar{B}([\frac{t(T)}{T}]) ,$$

where we have denoted time index $t(T)$ as a function of T . Next by using the continuous mapping theorem and the functional central limit theorem (see Billingsley (1967) for instance), we have the weak convergence as

(A.7) $$\frac{1}{T^{d+D+\frac{1}{2}}} \sum_{t=1}^{T} \mathbf{z}_t u_t = \frac{1}{T} \sum_{t=1}^{T} \mathbf{z}_t [\frac{1}{T^d n^{D-1}\sqrt{n}} u_t][\frac{1}{s}]^{D-\frac{1}{2}}$$
$$\stackrel{w}{\Longrightarrow} (\frac{1}{s})^{D-\frac{1}{2}} \int_0^1 \mathbf{z}(r)\bar{B}(r)dr .$$

Also by using the convergence condition on $\{\mathbf{z}_t\}$ as

(A.8) $$\frac{1}{T} \sum_{t=1}^{T} \mathbf{z}_t \mathbf{z}_t^{'} = (\frac{1}{s}) \sum_{i=1}^{s} \frac{1}{n} \sum_{j=1}^{n} \mathbf{z}_{(j-1)s+i} \mathbf{z}_{(j-1)s+i}^{'}$$
$$\longrightarrow \mathbf{M}^* = \frac{1}{s} \sum_{i=1}^{s} \mathbf{M}_i > 0 ,$$

and cheking the tightness condition, we have the weak convergence

$$\text{(A.9)} \qquad \frac{1}{T^{d+D-\frac{1}{2}}}(\hat{\beta}_{LS}-\beta) = (\frac{1}{T}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{z}_t^{'})^{-1}[\frac{1}{T^{d+D+\frac{1}{2}}}\sum_{t=1}^{T}\mathbf{z}_t u_t$$

$$\stackrel{w}{\Longrightarrow} (\frac{1}{s})^{D-\frac{1}{2}}\mathbf{M}^{-1}\int_0^1 \mathbf{z}(r)\bar{B}(r)dr \ ,$$

where $\hat{\beta}_{LS}=(\hat{\beta}_k)$ is the least squares estimator of the unknown vector $\beta=(\beta_k)$. Then by using similar and tedious arguments, we also have

$$\text{(A.10)} \qquad (\frac{1}{T})^{2(d+D)}\sum_{t=1}^{T}u_t^2 = (\frac{1}{s})^{2(D-\frac{1}{2})}\frac{1}{T}\sum_{t=1}^{T}[\frac{1}{T^d n^{D-1}\sqrt{n}}u_t]^2]$$

$$\stackrel{w}{\Longrightarrow} (\frac{1}{s})^{2(D-\frac{1}{2})}\int_0^1 \bar{B}^2(r)dr \ ,$$

and

$$\text{(A.11)} \qquad (\frac{1}{T})^{2(d+D)}\sum_{t=1}^{T}\hat{u}_t^2$$

$$\stackrel{w}{\Longrightarrow} (\frac{1}{s})^{2(D-\frac{1}{2})}[\int_0^1 \bar{B}^2(r)dr - (\int_0^1 \mathbf{z}(r)\bar{B}(r)dr)^{'}\mathbf{M}^{-1}(\int_0^1 \mathbf{z}(r)\bar{B}(r)dr \ ,$$

where $\hat{u}_t$ $(t=1,\cdots,T)$ are the least squares residuals. By gathering the above random terms, we finally have the desired weak convergence as

$$\text{(A.12)} \qquad \frac{1}{\sqrt{T}}t(\beta_k) \stackrel{w}{\Longrightarrow} t_k^* \ .$$

When $d=0$, we need to modify the above derivations slightly. For instance, (A.5) should be replaced by the simple form as

$$(A.5)^{'} \qquad u_t \sim u_t^* = \sum_{[\frac{t(T)}{s}]\geq s_1\geq\cdots\geq s_D\geq 1} v_{(s_D-1)s+i}$$

for $t=([\frac{t(T)}{s}]-1)s+i$ $(i=1,\cdots,s)$ and then (A.7) should be replaced by

$$(A.7)^{'} \qquad \frac{1}{T^{D+\frac{1}{2}}}\sum_{t=1}^{T}\mathbf{z}_t u_t \stackrel{w}{\Longrightarrow} (\frac{1}{s})^{D+\frac{1}{2}}\sum_{i=1}^{s}\int_0^1 \mathbf{z}_i(r)\bar{B}_i(r)dr \ .$$

Because the following derivations are completely pararell to those for the case when $d\geq 1$, we omit their details.

The rest of our proof is the results of standard and tedious arguments of weak convergence for weakly dependent time series which are routine. In the general case we divide both the numerator and the denominator of (2.11) by σ and also we need to show that we can ignore the initial conditions asymptotically. Then under the assumptions we have made it is straightforward to show that $\beta_0=1$ and $\beta_j=O(\rho^j)$ $(|\rho|<1)$, where we write

$$\text{(A.13)} \qquad u_t = \sum_{t\geq s_1\geq\cdots\geq s_{d-1}}\sum_{i=1}^{s}\sum_{[\frac{s_{d-1}}{s}]\geq s_d\geq\cdots\geq s_{d+D}\geq 1} u_{(s_{d+D}-1)s+i}^{(d+D)} \ ,$$

and

$$\text{(A.14)} \qquad u^{(d+D)}_{(s_{d+D}-1)s+i} \sim u^{(d+D)*}_{(s_{d+D}-1)s+i} = \sum_{j=0}^{\infty} \beta_j v_{(s_{d+D}-1)s+i-j} \; .$$

Hence we can show that the effects of weak dependence are cancelled out in the numerator and the denominator of the t-ratio and we have the desired result in the general case. *Q.E.D.*

Derivation of Theorem 3.2 :
Without loss of generality we take the seasonal dummy variables and we write $\sum_{j=1}^{s} \beta_j z_{jt} = \sum_{j=1}^{s} a_j I_t^{(j)}$, where $I_t^{(j)}$ $(j = 1, \cdots, s)$ are seasonal indicator functions. (If there are no seasonal dummy variables, we need to define the state vector and other notations accordingly.) Also we consider the case when $p \geq 1, P \geq 1, r \geq s$ and $1 \leq p \leq s$ for the resulting simplicity. (When $s \leq p \leq 2s$, we can use similar arguments below by constructing a different set of $2s + (p - s)$ dimensional vector and transformations involved, for instance. Nonetheless, the essential arguments are the same in other cases.) Then the (s+p)-dimensional state vector $\{Y_j\}$ has the vector representation

$$\text{(A.15)} \quad Y_j = [\mathbf{I}_{s+p} - \mathbf{D_0}]\mathbf{a} + \mathbf{D}_0 Y_j + \mathbf{D_1} Y_{j-1} + \sum_{i=2}^{P} \mathbf{B}_i^* Y_{j-i} - \sum_{i=1}^{P} \mathbf{A}_i Y_{j-i} + V_j,$$

where $V_j^{'} = (v_{(j-1)s+s}, \cdots, v_{(j-2)s+s-p+1})$ are $(s+p) \times 1$ random vectors, $\mathbf{a}^{'} = (a_s, \cdots, a_{s-p+1})$ is a $(s + p) \times 1$ vector, $\mathbf{B}_j = diag(\Phi_j^i)$ $(j = 1, \cdots, P)$ are $s \times s$ diagonal matrices,

$$\boldsymbol{B}_j^* = \begin{pmatrix} \mathbf{B}_j & O \\ \mathbf{O} & O \end{pmatrix} ,$$

$$\text{(A.16)} \qquad \boldsymbol{D}_0 = \begin{pmatrix} 0 & \phi_1 & \cdots & \phi_p & 0 & \cdots & & 0 \\ 0 & 0 & \phi_1 & \cdots & \phi_p & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & & & \\ 0 & \cdots & & & 0 & \phi_1 & \cdots & \phi_p \\ & O & O & \cdots & & & O & O \end{pmatrix} ,$$

$$\boldsymbol{A}_i = \begin{pmatrix} 0 & \phi_1\Phi_i^{s-1} & \cdots & \phi_p\Phi_i^{s-p} & 0 & \cdots & 0 \\ 0 & 0 & \phi_1\Phi_i^{s-2} & \cdots & \phi_p\Phi_i^{s-p-1} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & \phi_1\Phi_i^{1} & \cdots & \phi_p\Phi_i^{s-p+2} & 0 \\ 0 & \cdots & & & \phi_1\Phi_i^{s} & \cdots & \phi_p\Phi_i^{s-p+1} \\ & O & O & & & O & \end{pmatrix},$$

and

$$\boldsymbol{D}_1 = \begin{pmatrix} \mathbf{B}_1 & & O \\ \mathbf{I}_p & O & O \end{pmatrix} ,$$

are $(s + p) \times (s + p)$ matrices.
This vector representation of time series model looks very complicated. However, we have the next key results on the associated characteristic equation after lengthy manipulations of related matrices. For the completeness of our discussions we give two lemmas on the determinants of the associated matrices. We note that Lemma A.1 is a simple consequence of Lemma A.2.

Lemma A.1 : *The associated characteristic equation for the vector AR process defined by (A.15) can be written as*

$$c(\lambda) = |\lambda^P[\mathbf{I}_j - \mathbf{D}_0] - \lambda^{P-1}[\mathbf{D_1} - \mathbf{A}_1] - \sum_{i=2}^{P} \lambda^{P-i}[\mathbf{B}_i^* - \mathbf{A}_i]| = 0 \; . \tag{A.17}$$

Then we have the equality that

$$c(\lambda) = \prod_{i=1}^{p}(\lambda - \rho_i^s) \prod_{i=1}^{s}[\lambda^P - \lambda^{P-1}\Phi_1^i - \cdots - \Phi_P^i] \; , \tag{A.18}$$

where ρ_i $(i = 1, \cdots, p)$ *are the solutions of the equation*

$$\lambda^p - \lambda^{p-1}\phi_1 - \cdots - \phi_p = 0 \; . \tag{A.19}$$

Lemma A.2 : *We have the simple relation on the determinants of the associated matrix as*

$$|\begin{pmatrix} 1 & -\phi_1 & \cdots & -\phi_p & 0 & \cdots & 0 \\ 0 & 1 & -\phi_1 & \cdots & -\phi_p & 0 & 0 \\ 0 & 0 & 1 & \cdots & & & \\ 0 & 0 & 0 & \cdots & \cdots & & \\ 0 & \cdots & & 1 & -\phi_1 & \cdots & -\phi_p \\ -\phi_p & \cdots & 0 & \lambda & -\lambda\phi_1 & \cdots & -\lambda\phi_{p-1} \\ \cdots & & & & & & \\ -\phi_2 & \cdots & & & & \lambda & -\lambda\phi_1 \\ -\phi_1 & \cdots & -\phi_p & \cdots & & 0 & \lambda \end{pmatrix}| \tag{A.20}$$

$$= \prod_{i=1}^{p}(\lambda - \rho_i^s) \;\; ,$$

where ρ_i $(i = 1, \cdots, p)$ *are the solutions of the characteristic equation of (A.19).*

Proof of Lemma A.2 :
We define a sequence of $p \times 1$ vectors $\{\mathbf{c}_j\}$ $(j = -p+2, \cdots, p)$ by the difference equation

$$\mathbf{c}_j = \phi_1\mathbf{c}_{j-1} + \cdots + \phi_p\mathbf{c}_{j-p} \; (j = 2, \cdots, s-p) \; , \tag{A.21}$$

and the initial conditions

$$(\mathbf{c}_{-p+2}, \cdots, \mathbf{c}_1) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \phi_p \\ 1 & 0 & \cdots & 0 & \phi_{p-1} \\ 0 & 1 & 0 & \cdots & \phi_{p-2} \\ \cdot & & 0 & & \\ 0 & \cdots & & & \phi_1 \end{pmatrix} \; . \tag{A.22}$$

Also we define $\mathbf{c}_s$ by

$$\mathbf{c}_s = \phi_1\mathbf{c}_{s-1} + \cdot + \phi_p\mathbf{c}_{s-p} + \begin{pmatrix} -\phi_{p-1}\lambda \\ \vdots \\ -\phi_1\lambda \\ \lambda \end{pmatrix} \tag{A.23}$$

and the last (p-1) vectors $\mathbf{c}_{s-1},\cdots,\mathbf{c}_{s-(p-1)}$ in the similar way. By applying a sequence of elementary transformations of matrices, we can evaluate the determinant (A.20) which is equivalent to the determinant of $p\times p$ matrix $(\mathbf{c}_{s-(p-1)},\cdots,\mathbf{c}_s)$. Then we use the relation

$$
\text{(A.24)}\quad (\mathbf{c}_{s-p+1},\cdots,\mathbf{c}_s)=\lambda\mathbf{I}_p-(\mathbf{c}_2,\cdots,\mathbf{c}_p)\begin{pmatrix} 0 & 0 & \cdots & 0 & \phi_p \\ 1 & 0 & \cdots & 0 & \phi_{p-1} \\ 0 & 1 & 0 & \cdots & \phi_{p-2} \\ \cdot & & 0 & & \\ 0 & \cdots & & & \phi_1 \end{pmatrix}^{s-p} .
$$

Finally by considering the determinant of (A.24) and using the initial conditions on $\{\mathbf{c}_j\}$, we have the desired result.

Proof of Theorem 3.2 :
By using Lemma A.1, we notice that the stability assumptions in Theorem 3.2 are equivalent to the conditions that the absolute values of all characteristic roots of (A.18) are less than one. Then we immediately obtain the desired result by using the standard arguments in the statistical time series analysis. (See Chapter 5 of Anderson (1971) for instance.)

Proof of Theorem 3.3 :
We first multiply $I_t^{(i)}$ to (3.4) and take the expectations. Under the stability conditions, we have

$$
\text{(A.25)}\quad \mu_i=E[y_tI_t^{(i)}]=\frac{a_i}{1-b_i}\quad (i=1,\cdots,s) .
$$

Then by using the Ergodic Theorem we have

$$
\text{(A.26)}\quad \frac{1}{T}\sum_{t=1}^{T}y_t=\frac{1}{s}\sum_{i=1}^{s}[\frac{1}{n}\sum_{j=1}^{n}y_{(j-1)s+i}]\xrightarrow{p}\frac{1}{s}\sum_{i=1}^{s}\mu_i=\bar{\mu}
$$

as $n\uparrow+\infty$. In the similar vein, by multiplying $I_t^{(i)}$ to y_t^2 and taking the expectations, we have

$$
\text{(A.27)}\quad E[y_t^2I_t^{(i)}]=\frac{1}{1-b_i^2}[a_i^2+\sigma_i^2+2a_ib_i\mu_i] .
$$

Hence

$$
\text{(A.28)}\quad \frac{1}{T}\sum_{t=1}^{T}y_t^2=\frac{1}{s}\sum_{i=1}^{s}[\frac{1}{n}\sum_{j=1}^{n}y_{(j-1)s+i}^2]\xrightarrow{p}\frac{1}{s}\sum_{i=1}^{s}[\frac{a_i^2+\sigma_i^2+2a_i\sigma_i\mu_i}{1-b_i^2}]
$$

as $n\uparrow+\infty$. Then by using the relation of $a_i=(1-b_i)\mu_i$ $(i=1,\cdots,s)$, we can calculate the probability limit as

$$
\begin{aligned}
\text{(A.29)}\quad \frac{1}{T}\sum_{t=1}^{T}(y_t-\bar{y})^2 &= \frac{1}{T}\sum_{t=1}^{T}y_t^2-\bar{y}^2 \\
&\xrightarrow{p} \frac{1}{s}\sum_{i=1}^{s}\frac{\sigma_i^2}{1-b_i^2}+\frac{1}{s}\sum_{i=1}^{s}\frac{a_i(a_i+2b_i\mu_i)}{1-b_i^2}-\bar{\mu}^2 \\
&= \frac{1}{s}\sum_{i=1}^{s}\frac{\sigma_i^2}{1-b_i^2}+\frac{1}{s}\sum_{i=1}^{s}(\mu_i-\bar{\mu})^2
\end{aligned}
$$

as $n \uparrow +\infty$. Also we notice that

$$\text{(A.30)} \qquad \frac{1}{T}\sum_{t=1}^{T}(y_{t-s}-\bar{y}_{-s})(y_t-\bar{y})$$

$$= \frac{1}{T}\sum_{t=1}^{T}(y_{t-s}-\bar{\mu})[\sum_{i=1}^{s}(a_i+b_iy_{t-s}+\sigma_i v_t)-\bar{\mu}]+o_p(1) \ .$$

Hence by using simple calculations, its probability limit can be written as

$$\text{(A.31)} \qquad \sum_{i=1}^{s}a_iE[\frac{1}{T}\sum_{t=1}^{T}(y_t-\bar{\mu})I_t^{(i)}]+\sum_{i=1}^{s}b_iE[\frac{1}{T}\sum_{t=1}^{T}(y_{t-s}-\bar{\mu})y_{t-s}I_t^{(i)}]$$

$$= \sum_{i=1}^{s}a_i(\frac{\mu_i-\bar{\mu}}{s})+\sum_{i=1}^{s}b_i[\frac{1}{s}\frac{a_i^2+\sigma_i^2+2a_ib_i\mu_i}{1-b_i^2}-\bar{\mu}\frac{\mu}{s}]$$

$$= \frac{1}{s}\sum_{i=1}^{s}\frac{b_i\sigma_i^2}{1-b_i^2}+\frac{1}{s}\sum_{i=1}^{s}(\mu_i-\bar{\mu})^2 \ ,$$

where we have used the relation $a_i = (1-b_i)\mu_i$ $(i = 1, \cdots, s)$. Then we have the desired result. *Q.E.D.*

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