CIRJE-F-98

Small Verifiability in Long-Term Relationships

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December 2000

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The First Version: September 1, 2000
This Version: November 28, 2000

Abstract

This paper investigates a *finitely* repeated game at the beginning of which players agree to write an explicit contract with *full* commitment. This contract conditions the budget-balancing vector of side payments on the complete history of their action choices. However, this history is *not always* verifiable to the court, and each player's liability is severely *limited*. We show, in the general case, that even if the probability of the complete history being verifiable is small and the amount of possible fines is negligible compared with the differences in long-run payoffs, every efficient payoff vector induced by an action profile, which Pareto-dominates a Nash equilibrium action profile, is *uniquely* and *exactly* implementable either in terms of perfect equilibrium or in terms of perfect iterative undominance. Moreover, we show, in a wide class of component games, that even if this probability is *close to* zero and the amount of fines is negligible compared with the differences *not only* in long-run payoffs *but also* in instantaneous payoffs, every efficient payoff vector induced by an action profile, which Pareto-dominates a Nash equilibrium action profile, is *uniquely* and *virtually* implementable either in terms of perfect equilibrium or in terms of perfect iterative undominance.

Key Words: Finitely Repeated Games with Side Payments, Small Verifiability, Limited Liability, Unique Implementation, Efficiency

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[+] I am very grateful to Stefan Worrall for his careful reading. All errors are mine.

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1. Introduction

This paper investigates a T times *finitely* repeated game with perfect monitoring and no discounting, in which, at the beginning of period 1, players agree to write an explicit long-term contract that assigns a budget-balancing vector of side payments to each complete history of players' action choices. This contract has *full commitment* power in that in every period players cannot breach and renegotiate it multilaterally. However, the complete history is *not always* verifiable to the court. According to the contract, at the end of period *T*, players pay or receive side payments if and only if the complete history can be verified to the court, while they pay or receive no side payments if this history cannot be verified. We assume that each player's liability is *limited* to an upper bound of fines that she can afford to pay. We also assume that the number of repetitions T is large enough.

Our objective, in this paper, is to clarify whether, in the game just described, efficiency is *uniquely* implementable either in terms of perfect equilibrium or in terms of *perfect iterative undominance*. Perfect iterative undominance is a solution concept that is defined by the backward induction technique. Unique implementation in terms of this solution concept is more restrictive than that in terms of perfect equilibrium. This paper provides sufficient conditions under which there exists a side payment contract such that in a finitely repeated game there exists either a unique perfect equilibrium or a unique perfect iteratively undominated strategy profile, and this strategy profile induces, exactly or virtually, a fully collusive, i.e., an efficient, payoff vector.

The paper is in two parts. In the first part, we consider a *trigger strategy profile*. According to this profile, players will continually choose an efficient action profile, which Pareto-dominates a Nash equilibrium action profile, as long as no one has deviated, while they will continually choose the Nash equilibrium action profile *after* someone has deviated. The payoff vector induced by the trigger strategy profile is *exactly* the same as the payoff vector induced by the efficient action profile. Based on this trigger strategy profile, we specify a side payment contract in which a player will be fined if and only if she is the *last* deviant from the trigger strategy profile. We show, in the general case, that under this fairly simple form of contract, *even if the probability of the complete history being verifiable is small and each player's liability is limited, efficiency is uniquely and exactly implementable either in terms of perfect equilibrium or in terms of perfect iterative undominance.* When the amount of possible fines multiplied by the probability of the complete history being verifiable is larger than the maximum difference in instantaneous payoffs, the trigger strategy profile will become, in a finitely repeated game with this side payment contract, the *unique* perfect iteratively undominated strategy profile. This result is permissive, because we need no restrictions on the component game other than the assumption that there exists a strict and less preferable Nash equilibrium action profile, and also because the amount of possible fines can be negligible compared with the differences in long-run payoffs.

In the second part of the paper, we introduce a *modified* version of the trigger strategy profile, according to which, during the first $T-\hat{t}$ periods, players will continually choose the efficient action profile as long as no one has deviated, while they will continually choose the Nash equilibrium action profile after someone has deviated. During the last $\hat{t}$ periods, players will continually choose an action profile, which is contingent on the number of *the first deviants during the first* $T-\hat{t}$ *periods,* and which is Pareto-dominated by the efficient action profile, as long as no one has deviated, while they will continually choose the Nash equilibrium action profile after someone has deviated. The payoff vector induced by this modified trigger strategy profile is *virtually* the same as the payoff vector induced by the efficient action profile. Based on this profile, we specify an alternative side payment contract, where by a player will be fined if and only if she is the last deviant from the modified trigger strategy profile *during the last* $\hat{t}$ *periods*.

From this, we are able to prove that, in a finitely repeated game with this side payment contract, the modified trigger strategy profile is the unique perfect iteratively undominated strategy profile (the unique perfect equilibrium) provided that the component game satisfies minor restrictions and there exists a unique iteratively undominated action profile (a unique Nash equilibrium action profile, respectively) in the component game. This result is again permissive, because we need no restrictions on the probability of the complete history being verifiable or on the upper bound of fines. Hence, we can conclude, in a wide class of component games, that *even if the probability of the complete history being verifiable is close to zero and the amount of fines is negligible compared with the differences not only in long-run payoffs but also in instantaneous payoffs, efficiency is uniquely and virtually implementable either in terms of perfect equilibrium or in terms of perfect iterative undominance.*

There exists a huge volume of previous works on repeated games that provide a theoretical foundation to the widely accepted view that in real economic situations, long-term relationships facilitate collusion more than do short-term relationships. These works commonly assume that Anti-Trust Law prohibits players from writing an explicit contract, or that it is impossible for players to enforce any history-contingent explicit contract because their action choices are unverifiable to the court. Fudenberg and

Maskin (1986) investigated infinitely repeated games and provided the Folk Theorem in terms of perfect equilibrium, in that every individually rational payoff vector, whether it is collusive or not, can be approximated by a perfect equilibrium payoff vector, when the discount factor is close to zero. Benoit and Krishna (1985) and Friedman (1985) investigated finitely repeated games and provided sufficient conditions under which the Folk Theorem or the efficiency result holds in terms of perfect equilibrium.[1] The reason why players can collude without the use of explicit contracts is put as follows. In every period players are confronted with the same subgame, and this subgame has multiple perfect equilibria. The logic behind collusive behavior depends crucially on this multiplicity. If a player deviates unilaterally from collusive behavior, her opponents will retaliate from the next period by playing an unfavorable equilibrium. Because of this future penalty induced by the move to the unfavorable equilibrium, each player hesitates to deviate and earn the instantaneous gain.

In sharp contrast to this orthodoxy, the present paper adopts an alternative basis for collusion to occur. Subgames of a finitely repeated game with side payments *differ* across past histories because the history-contingent contract influences the payoff structures of these subgames. Moreover, these subgames each have their own respective *unique* perfect equilibria. If a player deviates unilaterally from collusive behavior, all players will be confronted in the next period with the subgame whose unique perfect equilibrium is unfavorable to the deviant. Because of this future penalty induced by the move to the unfavorable subgame, each player hesitates to deviate and earn the instantaneous gain.

Because, in a finitely repeated game with side payments, collusive behavior can be described as a *unique* perfect equilibrium, the predictive power of the results in this paper is much stronger than that of the Folk Theorem. More precisely, the fact that not only the whole game but also every subgame satisfies this uniqueness, implies that, in every period, players have *no* room to renegotiate the terms of *implicit* agreement multilaterally and improve their welfare. This point contrasts with the fact that renegotiation-proofness on the terms of implicit agreement has long been a controversial issue in the literature of implicit collusion.[2]

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[1] For further references, see Pearce (1992). Radner (1980), Chu and Geanapoplos (1988), and Conlon (1996) investigated finitely repeated games, and provided their respective efficiency results, on the assumption that players are slightly irrational.

[2] See, for example, Bernheim and Rey (1989), Farrell and Maskin (1989) and Pearce (1987) on infinitely repeated games, and Benoit and Krishna (1988) on finitely repeated games.

The present paper is also in contrast with the literature of the reputational theory of finitely repeated games, which assumes incomplete information on players' types.[3] Several works such as Kreps and Wilson (1982) and Milgrom and Roberts (1982) provided their respective examples of a chain-store game in which there exists a unique perfect Bayesian equilibrium and it virtually induces an efficient allocation. However, these results depend crucially on their own specifications of the incomplete information structure, while it is hard to tell about how players can determine the well-behaved incomplete information structure in advance of their repeated play. The present paper, however, assumes complete information and is based on a more plausible scenario of a contracting process, in which players will collectively agree to a side payment contract that guarantees both uniqueness and, exact or virtual, efficiency.[4]

There also exists a sizeable literature dealing with the agency problem with moral hazard that specifically seeks to clarify whether a single-period relationship attains the first-best or second-best allocation through the writing of explicit contracts.[5] This literature commonly makes the assumption that it is difficult for the court to verify players' action choices, but that there exists a public signal that is randomly determined according to a probability distribution conditional on player's action choices, and this signal *is* verifiable. Hence, players can agree to write an arbitrary explicit contract that conditions the budget-balancing vector of side payments not on their action choices but on the realization of this signal. A large proportion of this literature is devoted to investigating the single-agent problem, while several works such as Holmstrom (1982), Williams and Radner (1989), and Legros and Matsushima (1991) investigate multi-agent relationships. Legros and Matsushima provided a necessary and sufficient condition under which there exists a side payment contract that induces players to choose a collusive action profile as a Nash equilibrium on the assumption that utilities

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[3] For the survey of this literature, see Fudenberg (1992).

[4] For example, one of the partners, called the principal, decides a side payment contract, which maximizes her own payoff given the constraint that there exists a unique perfect equilibrium and that this perfect equilibrium induces a payoff vector that is no worse than the payoff vector that partners can receive outside if they fail to establish their partnership. Another scenario is that all players agree to a side payment contract according to the Nash bargaining solution at the beginning of period 1 given the constraint that there exists a unique perfect equilibrium and that the payoff vector that partners can receive outside is regarded as the threat point.

[5] For the references, see the surveys by Hart and Holmstrom (1987), Dutta and Radner (1994), and Salanie (1997).

are quasi-linear.[6] This side payment contract, however, requires that players pay a *large* fine as penalty for deviation.

In contrast, the present paper assumes that players establish a long-term relationship, but that only small fines, which may be close to zero, exist in totality. The paper shows that the establishment of a long-term relationship, together with the use of a contract that fines only the last deviants, dramatically economizes on monetary fines without harming players' incentive to collude.

As such, this paper may offer an important economic implication within the field of law and economics. In real situations of labor contracting with moral hazard, it is practically difficult to establish measures of performance that are always verifiable to the court. It is also unrealistic to expect that a limitedly liable worker will be able to pay a large fine when the fact that she has neglected her duty could be disclosed to the public. From this, it is widely believed that in real situations the legal enforcement of explicit contracts plays only a limited role in resolving issues of moral hazard. Many economists, such as MacLeod and Malcomson (1989), have emphasized that the self-enforcement of implicit contracts instead plays a more crucial role than legal enforcement, which are thought of formally as a perfect equilibrium in an infinitely repeated game. In contrast, the present paper shows that even if workers' performances are hardly verifiable and their liability is severely limited, the role of legal enforcement is very crucial and even indispensable for workers' incentives.

A technical aspect of this paper is related to Abreu and Matsushima (1992) and Glazer and Perry (1996), which investigated the implementation of social choice functions in complete information environments and showed that every social choice function is virtually implementable either in terms of iterative undominance or in terms of perfect equilibrium. Both works construct mechanisms in which only small fines are used. Abreu and Matsushima constructed a simultaneous-move mechanism, while Glazer and Perry constructed, as a modification of the Abreu-Matsushima mechanism, an alternating-move mechanism in which only the last deviants will be fined. Without the use of fines, there exist multiple perfect equilibria, one of which is the socially optimal strategy profile. Hence, both works suggest that the use of small fines can eliminate all unwanted equilibria, whether players' messages are hidden or not.

In contrast, the present paper depends heavily on the assumption that players' action choices are not hidden, because the strategy profile that players want to enforce depends

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[6] These works did not consider the uniqueness of equilibrium. Ma (1988), and Ma, Moore and Turnbull (1988) investigated unique implementation in multi-agent relationships.

on the histories of their action choices. Moreover, without the use of fines this strategy profile may not satisfy the perfect equilibrium property. Hence, the use of small fines in this paper is regarded as playing the crucial role not only of eliminating the unwanted equilibria but also of describing the wanted strategy profile as an equilibrium.

The organization of this paper is as follows. Section 2 presents the model. Section 3 corresponds to the first part of this paper, which investigates exact implementation. Section 4 corresponds to the second part, which investigates virtual implementation. Section 5 investigates side payment contracts other than those constructed in Sections 3 and 4. Section 6 concludes.

2. Finitely Repeated Games with Side Payments

A finitely repeated game with side payments $\Gamma = \Gamma(T, N, (A_i, u_i)_{i\in N}, M, m, \varepsilon)$ is defined as follows. The component game is given by $G \equiv (N, (A_i, u_i)_{i\in N})$, where $N = \{1,...,n\}$ is the finite set of players, A_i is the set of actions for player $i \in N$, $A \equiv \prod_{i\in N} A_i$, and $u_i : A \to R$ is the instantaneous payoff function for player i. We assume that there exists a *strict* Nash equilibrium action profile in G, which is denoted by $a^e \in A$.

Let Q_i denote an arbitrary nonempty subset of A_i, and let $Q \equiv \prod_{i\in N} Q_i$. An action $a_i \in A_i$ for player i is said to be dominated w. r. t. Q if $a_i \in Q_i$ and there exists $a_i' \in Q_i / \{a_i\}$ such that $v_i(a / a_i') > u_i(a)$ for all $a_{-i} \in Q_{-i}$. Let $\tilde{Q}_i(Q)$ denote the set of all undominated actions for player i w. r. t. Q. Let $Q_i^0 \equiv A_i$, $Q^k \equiv \prod_{i\in N} Q_i^k$, $Q_i^k \equiv \tilde{Q}_i(Q^{k-1})$, and $Q^* \equiv \bigcap_{k=0}^{\infty} Q^k$. An action profile $a \in A$ is said to be *iteratively undominated* in G if $a \in Q^*$. Note that the strict Nash equilibrium action profile a^e is iteratively undominated in G.

We denote by a positive integer $T > 0$ the number of periods in which players repeatedly play G. A history up to period $t = 1,...,T$ is denoted by $h(t) \equiv (a(1),\ldots,a(t))$, where $a(\tau) \equiv (a_1(\tau),\ldots,a_n(\tau)) \in A$ is the action profile chosen in period τ. A history up to period T is called a *complete* history. Let $h(0)$ denote the *null* history. The set of histories up to period t is denoted by $H(t)$. We assume that monitoring is *perfect*, i.e., players can observe the other players' action choices at the end of every period. Hence, in every period $t = 1,...,T$, the realized history $h(t-1) \in H(t-1)$ is always common knowledge among players.

We, however, assume that monitoring is *hardly* verifiable to the court. The complete history $h(T) \in H(T)$ can (cannot) be verified to the court with probability ε (probability $1-\varepsilon$, respectively) at the end of period *T*. We assume that ε is positive but close to zero.

We assume that side payments among players are possible. At the beginning of period 1, players agree to write a *side payment contract*, which is contingent on the complete history. The side payment contract has *full commitment* power in that in every period players cannot breach and renegotiate it multilaterally. We assume that there exists an upper bound to the amount of fines, $M > 0$, such that each player is able to pay up to this amount in totality. A side payment contract is defined by $m = (m_1,...,m_n)$, where for every $i \in N$,

$$m_i : H(T) \to [-M, \infty),$$

and m is *budget-balancing*, i.e.,

$$\sum_{i \in N} m_i(h(T)) = 0 \quad \text{for all} \quad h(T) \in H(T).$$

According to the side payment contract m, each player $i \in N$ receives side payment $m_i(h(T))$ at the end of period T if $h(T)$ can be verified to the court, whereas she pays or receives no side payment if $h(T)$ cannot be verified. Hence, the *expected* long-run payoff for player i, given that the complete history $h(T) = (a(1), \dots, a(T)) \in H(T)$ occurs, is defined by

$$v_i(h(T)) = v_i(h(T); \Gamma) \equiv \sum_{t=1}^{T} u_i(a(t)) + \varepsilon m_i(h(T)),$$

where we assume no discounting.[7] Let $v(h(T)) \equiv (v_1(h(T)), \dots, v_n(h(T)))$. For simplicity, we will denote Γ or $\Gamma(T, m)$ instead of $\Gamma = \Gamma(T, N, (A_i, u_i)_{i \in N}, M, m, \varepsilon)$ whenever there exists no confusion.

Given the number of repetitions T, a strategy for player i is defined by $s_i : \bigcup_{t=0}^{T-1} H(t) \to A_i$. Let S_i denote the set of strategies for player i. Let $S \equiv \prod_{i \in N} S_i$ and $s(h(t)) \equiv (s_1(h(t)), \dots, s_n(h(t))) \in A$. For every $t = 0, \dots, T-1$, every $h(t) \in H(t)$, and every strategy profile $s = (s_1, \dots, s_n) \in S$, the complete history induced by $(s, h(t))$ is defined by $h(s, h(t)) = (h(t), a(t+1), \dots, a(T)) \in H(T)$, where

$$a(t+1) = s(h(t)),$$

$$h(t+1) = (h(t), a(t+1)),$$

and for every $\tau = t+2, \dots, T$,

$$a(\tau) = s(h(\tau - 1)),$$

and

$$h(\tau) = (h(\tau - 1), a(\tau)).$$

A strategy profile $s \in S$ is said to be a *perfect equilibrium in* Γ if for every $t = 0, \dots, T-1$, every $h(t) \in H(t)$, and every $i \in N$,

$$v_i(h(s, h(t))) \geq v_i(h(s / s_i', h(t))) \quad \text{for all} \quad s_i' \in S_i. \qquad (1)$$

Let $v_i(s) \equiv v_i(h(s, h(0)))$ and $v(s) \equiv (v_1(s), \dots, v_n(s))$.

We introduce another solution concept referred to as *perfect iterative undominance*, which is defined by the backward induction technique as follows. Let W_i denote an

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[7] We can extend this model to the case where players discount their future payoffs but the discount factor is close to unity. This paper assumes that players' instantaneous payoff functions are quasi-linear. We can also extend this model to the case where these instantaneous payoff functions are increasing w. r. t. players' own side payments but are not quasi-linear.

arbitrary subset of S_i, and let $W \equiv \prod_{i \in N} W_i$. A strategy $s_i \in S_i$ for player i is said to be dominated w. r. t. W given $h(t-1)$, if $s_i \in W_i$ and there exists $s_i' \in W_i$ such that

$$v_i(h(s/s_i', h(t-1))) > v_i(h(s, h(t-1))) \text{ for all } s_{-i} \in W_{-i}.$$

Let $\tilde{W}_i(W, h(t-1))$ denote the set of all undominated strategies for player i w. r. t. W given $h(t-1)$. Let

$$W_i^0(h(T-1)) \equiv S_i,$$
$$W^k(h(T-1)) \equiv \prod_{i \in N} W_i^k(h(T-1)),$$
$$W_i^k(h(T-1)) \equiv \tilde{W}_i(W^{k-1}(h(T-1)), h(T-1)),$$
$$W_i^\infty(h(T-1)) \equiv \bigcap_{k=0}^{\infty} W_i^k(h(T-1)),$$

and

$$W_i^*(T-1) \equiv \bigcap_{h(T-1) \in H(T-1)} W_i^\infty(h(T-1)).$$

Recursively, for every $t \in \{0, \ldots, T-2\}$, let

$$W_i^0(h(t)) \equiv W_i^*(t+1),$$
$$W^k(h(t)) \equiv \prod_{i \in N} W_i^k(h(t)),$$
$$W_i^k(h(t)) \equiv \tilde{W}_i(W^{k-1}(h(t)), h(t)),$$
$$W_i^\infty(h(t)) \equiv \bigcap_{k=0}^{\infty} W_i^k(h(t)),$$
$$W^\infty(h(t)) \equiv \prod_{i \in N} W_i^\infty(h(t)),$$
$$W_i^*(t) \equiv \bigcap_{h(t) \in H(t)} W_i^\infty(h(t)),$$

and

$$W^*(t) \equiv \prod_{i \in N} W_i^*(t).$$

A strategy profile $s \in S$ is said to be *perfect iteratively undominated* in Γ if $s \in W^*(0)$. Note that if a strategy profile $s \in S$ is perfect iteratively undominated in Γ, then it satisfies the perfect iterative undominance property in every subgame. Note that a strategy profile is a perfect equilibrium in Γ if it is perfect iteratively undominated in Γ, but that the reverse is not true.

Throughout this paper, we shall fix an action profile $a^* \in A$, which may induce an efficient payoff vector. We assume that this action profile a^* *Pareto-dominates* the Nash equilibrium action profile a^e in G, i.e.,

$$u(a^*) > u(a^e).$$

For every $s \in S$, and every $h(T) \in H(T) / \{h(s, h(0))\}$, we define a period $\tilde{t} = \tilde{t}(s, h(T)) \in \{1, ..., T\}$ by

$$a(\tilde{t}) \neq s(h(\tilde{t}-1)),$$

and

$$a(t) = s(h(t-1)) \quad \text{for all} \quad t \in \{\tilde{t} + 1, \ldots, T\}.$$

Period $\tilde{t}(s, h(T))$ is regarded as the *last* period in which the chosen action profile was not the same as that suggested by strategy profile s.

3. Exact Implementation

Given an arbitrary number of repetitions $T \geq 1$, we specify a trigger strategy profile $s^{+} \in S$ as follows. For every $t \in \{1,\ldots,T\}$, and every $h(t-1) \in H(t-1)$,

$$s^{+}(h(t-1)) = a^{*} \quad \text{if} \quad a(\tau) = a^{*} \quad \text{for all} \quad \tau < t,$$

and

$$s^{+}(h(t-1)) = a^{e} \quad \text{if} \quad a(\tau) \neq a^{*} \quad \text{for some} \quad \tau < t.$$

According to the trigger strategy profile s^{+}, in each period, players choose a^{*} if they have chosen a^{*} in all previous periods, while they choose a^{e} otherwise.

We denote by $n(a,a') \in \{0,\ldots,n\}$ the number of players $i \in N$ satisfying that $a'_i \neq a_i$. Based on the trigger strategy profile s^{+}, we specify a side payment contract m^{+} as follows. For every $i \in N$,

$$m_i^{+}(h(s^{+},h(0))) = 0,$$

for every $h(T) \in H(T)/\{h(s^{+},h(0))\}$, and for $\tilde{t} = \tilde{t}(s^{+},h(T))$,

$$m_i^{+}(h(T)) = \left(\frac{n(a(\tilde{t}),s^{+}(\tilde{t}-1))-1}{n-1}-1\right)M \quad \text{if} \quad a_i(\tilde{t}) \neq s_i^{+}(h(\tilde{t}-1)),$$

and

$$m_i^{+}(h(T)) = \frac{n(a(\tilde{t}),s^{+}(\tilde{t}-1))}{n-1}M \quad \text{if} \quad a_i(\tilde{t}) = s_i^{+}(h(\tilde{t}-1)).$$

Here, $n(a(\tilde{t}),s^{+}(\tilde{t}-1))$ implies the number of the last deviants from s^{+} in $h(T) \in H(T)$. According to the side payment contract m^{+}, *a player will be fined if and only if she is the last deviant from the trigger strategy profile* s^{+}. Note that m^{+} is budget-balancing. Note that

$$\frac{1}{T}v(h(s^{+},h(0))) = u(a^{*}),$$

and, therefore, the strategy profile s^{+} induces *exactly* the same payoff vector as $u(a^{*})$ in $\Gamma(T,m^{+})$, irrespective of $T \geq 1$.

The following theorem provides sufficient conditions under which s^{+} is either the unique perfect equilibrium or the unique perfect iteratively undominated strategy profile in $\Gamma(T,m^{+})$, and therefore, *the payoff vector* $u(a^{*})$ *is uniquely and exactly implementable either in terms of perfect equilibrium or in terms of perfect iterative undominance, irrespective of* $T \geq 1$. For every $a \in A$, we define the component game $G^{[a]} = (N,(A_i,u_i^{[a]})_{i \in N})$ by

$$u_i^{[a]}(a') = u_i(a') - \varepsilon M \quad \text{if} \quad a'_i \neq a_i,$$

and

$$u_i^{[a]}(a') = u_i(a') \text{ if } a_i' = a_i .$$

In the component game $G^{[a]}$, each player receives the same instantaneous payoff as that in G, but she will be fined by the amount of M with probability ε if and only if she deviates from action profile a.

Theorem 1: *If a^e is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in $G^{[a^e]}$ and a^* is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in $G^{[a^*]}$, then s^+ is the unique perfect equilibrium (the unique perfect iteratively undominated strategy profile, respectively) in $\Gamma(T, m^+)$.*

Proof: See the Appendix.

Note that a^e is the unique iteratively undominated action profile in $G^{[a^e]}$ if

$$M > \frac{u_i(a) - u_i(a / a_i^e)}{\varepsilon} \text{ for all } a \in A \text{ and all } i \in N , \quad (2)$$

and that a^* is the unique iteratively undominated action profile in $G^{[a^*]}$ if

$$M > \frac{u_i(a) - u_i(a / a_i^*)}{\varepsilon} \text{ for all } a \in A \text{ and all } i \in N . \quad (3)$$

Hence, it follows that if the upper bound M is large enough to satisfy inequalities (2) and (3), then s^+ is the unique perfect iteratively undominated strategy profile in $\Gamma(T, m^+)$, and therefore, the payoff vector $u(a^*)$ is uniquely and exactly implementable in terms of perfect iterative undominance.

More specifically, note that inequalities (2) and (3) hold and, therefore, $u(a^*)$ is uniquely and exactly implementable in terms of perfect iterative undominance, if the amount of possible fines multiplied by the probability of the complete history being verifiable is larger than the maximum difference in instantaneous payoffs, i.e.,

$$\varepsilon M > \max_{i \in N, a \in A, a' \in A} [u_i(a) - u_i(a')].$$

Note also that the sufficient conditions in Theorem 1, or inequalities (2) and (3), do not depend on T, and therefore, given that M satisfies these conditions, $\frac{M}{T}$ tends towards zero as T increases. Hence, we can regard the upper bound M as a negligible amount compared with the differences in long-run payoffs, provided that the number of repetitions T is large enough.

The following example is useful for capturing the logical essence of Theorem 1.

Example 1: Consider an example of the component game G with two players, which is described by Figure 1.

[Figure 1: Component Game G]

Note that there exist two strict Nash equilibrium action profiles in G, i.e., (d,d) and (d',d'), and that the action profile (c,c) is efficient. Let $a^* = (c,c)$ and $a^e = (d,d)$, where

$$u(a^*) = (1,1) > (0,0) = u(a^e).$$

Moreover, let

$$\varepsilon M = 6.$$

The component games $G^{[(c,c)]}$ and $G^{[(d,d)]}$ are described by Figures 2 and 3, respectively.

[Figure 2: Component Game $G^{[(c,c)]}$]

[Figure 3: Component Game $G^{[(d,d)]}$]

Note that (c,c) and (d,d) are the *strictly dominant* action profiles in $G^{[(c,c)]}$ and $G^{[(d,d)]}$, respectively. Note also that action profile (d',d'), which is a strict Nash equilibrium action profile in G, and which is not Pareto-dominated by (c,c), is not a Nash equilibrium action profile either in $G^{[(c,c)]}$ or in $G^{[(d,d)]}$.

Consider an arbitrary period $t = 1,...,T$ and an arbitrary history $h(t-1) \in H(t-1)$. Suppose that players will behave according to the trigger strategy profile s^+ after period $t+1$. We show below that in period t, players will choose a strictly dominant action profile, which is the same as the one suggested by the trigger strategy profile s^+.

First, suppose that there exists a player who has deviated in previous periods. Then, the strategy profile s^+ suggests to players to choose (d,d) in period t, and, according to the side payment contract m^+, any player who does not choose action d in period t, will be fined, because players will behave according to s^+ after period $t+1$. Since players will continually choose the same action profile (d,d) after period $t+1$ irrespective of their action choices in period t, it follows that the difference in long-run payoffs for each player $i = 1,2$ between the choice of action d and the choice of any action $a_i(t)$ other than action d is equal to

$u_i(d, a_j(t)) - u_i(a_i(t), a_j(t)) + \varepsilon M$ if either $a_j(t) \neq d$ or both players are the last players who have deviated in previous periods,[8]

$u_i(d, a_j(t)) - u_i(a_i(t), a_j(t)) + 2\varepsilon M$ if $a_j(t) = d$ and player i is not the last player who has deviated in previous periods, i.e., the other player j is the single last player who has deviated in previous periods,

and

$u_i(d, a_j(t)) - u_i(a_i(t), a_j(t))$ if $a_j(t) = d$ and player i is the single last player who has deviated in previous periods,

where $j \neq i$. In the first and second cases, player i will not be fined as long as she chooses action d in period t, because the other player j will be the last deviant. Hence, in the first and second cases, player i is confronted with her own payoff structure that is essentially the same as the one of the component game $G^{[(d,d)]}$, and, in the second case, player i, by choosing action d in period t, additionally receives an expected transfer $\varepsilon M = 6$ from player j. In the third case, player i will be fined irrespective of her own choice in period t, because she will be the single last deviant irrespective of her own choice in period t. Hence, in the third case, player i is confronted with her own payoff structure that is essentially the same as the one of the component game G. Since (d, d) is a strict Nash equilibrium action profile in G and is strictly dominant in $G^{[(d,d)]}$, these differences in long-run payoffs for player i are all larger than zero. Hence, the choice of action d, which is the same as the one suggested by s^+, is strictly dominant in period t.

Next, suppose that there exists no player who has deviated in previous periods. Then, the strategy profile s^+ suggests to players to choose (c, c), and, according to the side payment contract m^+, any player who does not choose action c in period t, will be fined, because players will behave according to s^+ after period $t+1$. Players will continually choose (d, d) after period $t+1$, if some player does not choose action c in period t, while players will continually choose (c, c) after period $t+1$ if they choose (c, c) in period t. Hence, the difference in long-run payoffs for each player $i = 1,2$ between the choice of action c and the choice of any action $a_i(t)$ other than action c is equal to

$$u_i(c, a_j(t)) - u_i(a_i(t), a_j(t)) + \varepsilon M \quad \text{if} \quad a_j(t) \neq c,$$

and

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[8] Here, $u_i(a_i, a_j)$ denotes the payoff for player i when player i chooses action a_i and the other player $j \neq i$ chooses action a_j.

$u_i(c, a_j(t)) - u_i(a_i(t), a_j(t)) + \varepsilon M + (T - t)\{u_i(c,c) - u_i(d,d)\}$ if $a_j(t) = c$,

where $j \neq i$. In both cases, player i is confronted with her own payoff structure that is the essentially same as the one of the component game $G^{[(c,c)]}$, and, in the latter case, by choosing action $a_i(t) \neq c$, player $i's$ payoff in every future period decreases from $u_i(c,c) = 1$ to $u_i(d,d) = 0$. Since (c,c) is strictly dominant in $G^{[(c,c)]}$, these differences in long-run payoffs for player i are all larger than zero. Hence, the choice of action c, which is the same as the one suggested by s^+, is strictly dominant in period t.

By applying the above arguments recursively from period $t = T$ to period $t = 1$, we have concluded, from the definition of perfect iterative undominance, that s^+ is the unique perfect iterative undominated strategy profile in $\Gamma(T, m^+)$ and, therefore, the efficient payoff vector $u(c,c) = (1,1)$ is uniquely and exactly implementable in terms of perfect iterative undominance.

4. Virtual Implementation

We now investigate the situation in which the upper bound M is so close to zero that it is a negligible amount compared with the differences not only in long-run payoffs *but also* in instantaneous payoffs.

Choose $n+1$ action profiles $(a^{(r)})_{r=0}^{n}$ arbitrarily, where we assume that

$$u(a^*) > u(a^{(0)}) > \cdots > u(a^{(n-1)}) > u(a^{(n)}) = u(a^e). \qquad (4)$$

Let $\hat{t} \geq 1$ denote an arbitrary positive integer satisfying that for every $i \in N$, every $r = 1,\ldots,n$, and every $a \in A$,

$$u_i(a / a_i^*) > u_i(a) - \hat{t}\{u_i(a^{(r-1)}) - u_i(a^{(r)})\}. \qquad (5)$$

For every $t \geq 1$, we denote by $H(t,0)$ the set of all histories $h(t) \in H(t)$ up to period t satisfying that

$$a(t') = a^* \text{ for all } t' \leq \min[t, T - \hat{t}].$$

For every $t \geq 1$, and every $r \in \{1,...,n\}$, we denote by $H(t,r)$ the set of all histories $h(t) \in H(t)$ up to period t satisfying that there exists $\tau \leq \min[t, T - \hat{t}]$ such that

$$a(t') = a^* \text{ for all } t' < \tau,$$

$$a(\tau) \neq a^*,$$

and

$$n(a(\tau), a^*) = r.$$

Here, τ is the *first* period in which the chosen action profile is not the same as a^*, and $n(a(\tau), a^*) = r$ implies the number of the *first* deviants from a^*. Hence, $H(t,r)$ is regarded as the set of all histories up to period t such that the number of the first deviants from a^* during the first $\min[t, T - \hat{t}]$ periods is equal to r. Let $H(0,0) \equiv H(0)$.

Given $(a^{(r)})_{r=0}^{n}$, we specify a modified trigger strategy profile $s^* \in S$ as follows. For every $t \in \{1,\ldots,T - \hat{t}\}$,

$$s_i^*(h(t-1)) = a^* \text{ if } h(t-1) \in H(t-1,0),$$

and

$$s_i^*(h(t-1)) = a^e \text{ if } h(t-1) \notin H(t-1,0).$$

For every $t \in \{T - \hat{t} + 1,\ldots,T\}$, every $r \in \{0,\ldots,n\}$, and every $h(t-1) \in H(t-1,r)$,

$$s_i^*(h(t-1)) = a^e \text{ if } a(\tau) \neq a^{(r)} \text{ for some } \tau \in \{T - \hat{t} + 1,\ldots,t-1\},$$

and

$$s_i^*(h(t-1)) = a^{(r)} \text{ if } a(\tau) = a^{(r)} \text{ for all } \tau \in \{T - \hat{t} + 1,\ldots,t-1\}.$$

According to the modified trigger strategy profile s^*, in every period $t \leq T - \hat{t}$, players choose a^* if they have chosen a^* in all previous periods, while they choose a^e

otherwise. In every period $t \geq T - \hat{t} + 1$, given that $r \in \{0, \ldots, n\}$ is the number of the first deviants from a^* during the first $T - \hat{t}$ periods, players choose $a^{(r)}$ if they have chosen $a^{(r)}$ in all previous periods after period $T - \hat{t} + 1$, while they choose a^e otherwise. Since $u(a^{(r)}) > u(a^{(r+1)})$ for every $r \in \{0, ..., n-1\}$, it follows that as the number of the first deviants from a^* during the first $T - \hat{t}$ periods increases, the play during the last $\hat{t}$ periods suggested by s^* becomes less favorable for all players.

Figures 4 and 5 describe the modified trigger strategy profile s^*.

[Figure 4: Modified Trigger Strategy Profile s^*: From Period 1 to Period $T - \hat{t}$]

[Figure 5: Modified Trigger Strategy Profile s^*: From Period $T - \hat{t} + 1$ to Period T]

Based on the modified trigger strategy profile s^*, we specify a side payment contract m^* as follows. For every $i \in N$,

$$m_i^*(h(s^*, h(0))) = 0,$$

and, for every $h(T) \in H(T) / \{h(s^*, h(0))\}$, and for $\tilde{t} = \tilde{t}(s^*, h(T))$,

$$m_i^*(h(T)) = 0 \quad \text{if} \quad \tilde{t} \leq T - \hat{t},$$

$$m_i^*(h(T)) = \left(\frac{n(a(\tilde{t}), s^*(h(\tilde{t} - 1))) - 1}{n - 1} - 1\right)M \quad \text{if} \quad \tilde{t} > T - \hat{t} \quad \text{and} \quad a_i(\tilde{t}) \neq s_i^*(h(\tilde{t} - 1)),$$

and

$$m_i^*(h(T)) = \frac{n(a(\tilde{t}), s^*(h(\tilde{t} - 1)))}{n - 1} M \quad \text{if} \quad \tilde{t} > T - \hat{t} \quad \text{and} \quad a_i(\tilde{t}) = s_i^*(h(\tilde{t} - 1)).$$

According to the side payment contract m^*, a player i will be fined if and only if she is the last deviant from the modified trigger strategy profile s^* *during the last* $\hat{t}$ *periods.* Note that

$$v_i(h(s^*, h(0))) = (T - \hat{t})u_i(a^*) + \hat{t}u_i(a^{(0)}),$$

and, therefore, the strategy profile s^* *virtually* induces the payoff vector $u(a^*)$, provided that the number of repetitions T is large enough.

The following theorem provides sufficient conditions under which the modified trigger strategy profile s^* is either the unique perfect equilibrium or the unique perfect iteratively undominated strategy profile in $\Gamma(T, m^*)$.

Theorem 2: *If for every* $r \in \{0,...,n\}$ *the action profile* $a^{(r)}$ *is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in* $G^{[a^{(r)}]}$, *then* s^* *is the unique perfect equilibrium (the unique perfect iteratively undominated strategy profile, respectively) in* $\Gamma(T, m^*)$.

Proof: See the Appendix.

The following proposition provides sufficient conditions under which there exist $n+1$ action profiles $(a^{(r)})_{r=0}^{n}$ such that inequalities (4) and the suppositions in Theorem 2 hold.

Proposition 3: *Suppose that* $A_i = [0,1]$ *and* u_i *is differentiable w. r. t.* $a \in A$ *for every* $i \in N$*, and that there exists* $\Delta = (\Delta_1,...,\Delta_n) \in R^n$ *such that*

$$\sum_{i \in N} (\frac{\partial u_1(a^e)}{\partial a_i},...,\frac{\partial u_n(a^e)}{\partial a_i})\Delta_i > 0, \quad (6)$$

and for every $i \in N$,

either $a_i^e < 1$ *and* $\Delta_i > 0$, *or* $a_i^e > 0$ *and* $\Delta_i < 0$.

Then, there will exist $n+1$ *action profiles* $(a^{(r)})_{r=0}^{n}$ *such that inequalities (4) hold and* $a^{(r)}$ *is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in* $G^{[a^{(r)}]}$ *for every* $r \in \{0,...,n\}$, *if* a^e *is the unique Nash equilibrium action profile (the unique iteratively undominated action profile, respectively) in* G.

Proof: See the Appendix.

Note that the supposition of Proposition 3 is irrelevant to $\varepsilon > 0$ and $M > 0$. Hence, we can conclude, from Theorem 2 and Proposition 3, that the supposition of Proposition 3 is sufficient to show that for every $\varepsilon > 0$, every $M > 0$, every $a^* \in A$, and every large enough T, $u(a^*)$ *is uniquely and virtually implementable in terms of perfect equilibrium (in terms of perfect iterative undominance), if* $u(a^*) > u(a^e)$ *and* a^e *is the unique Nash equilibrium action profile (the unique iteratively undominated action profile, respectively) in* G.

Note that it is sufficient for the supposition of Proposition 3 that for every $i \in N$,

$$A_i = [0,1],\ a_i^e < 1,\ \frac{\partial u_i(a^e)}{\partial a_i} = 0,$$

and

$$\frac{\partial u_j(a)}{\partial a_i} > 0 \text{ for all } a \in A \text{ and all } j \neq i.$$

The following example is useful for capturing the logical essence of Theorem 2.

Example 2: Consider an example of the component game G with two players satisfying that for each $i = 1,2$,

$$A_i = [0,1],$$

and

$$u_i(a) = 2a_j - a_i \text{ for all } a \in A,$$

where $j \neq i$. Note that the action profile $(0,0)$ is strictly dominant in G and the action profile $(1,1)$ is efficient in G. Let $a^* = (1,1)$ and $a^e = (0,0)$. Fix $\varepsilon > 0$ and $M > 0$ arbitrarily, which are both close enough to zero to satisfy that

$$\varepsilon M < 1.$$

Let

$$a^{(0)} = (\frac{\varepsilon M}{2}, \frac{\varepsilon M}{2}), \quad a^{(1)} = (\frac{\varepsilon M}{4}, \frac{\varepsilon M}{4}), \text{ and } a^{(2)} = (0,0).$$

Note that inequalities (4) hold, i.e.,

$$u(a^*) = (1,1) > u(a^{(0)}) = (\frac{\varepsilon M}{2}, \frac{\varepsilon M}{2}) > u(a^{(1)}) = (\frac{\varepsilon M}{4}, \frac{\varepsilon M}{4})$$
$$> u(a^{(2)}) = (0,0),$$

and that $a^{(r)}$ is strictly dominant in $G^{[a^{(r)}]}$ for every $r \in \{0,1,2\}$. Choose an arbitrary positive integer $\hat{t}$, which is large enough to satisfy that

$$\hat{t} > \frac{4}{\varepsilon M}. \qquad (7)$$

Choose the number of repetitions T arbitrarily, which is large enough to satisfy that $T > \hat{t}$.

First, consider period $T - \hat{t} + 1$ and an arbitrary history $h(T - \hat{t}) \in H(T - \hat{t})$. Note that there exists $r \in \{0,1,2\}$ such that $h(T - \hat{t}) \in H(T - \hat{t}, r)$. The strategy profile s^* suggests to players to choose action profile $a^{(r)}$ in every period after period $T - \hat{t} + 1$ as long as players have continually chosen $a^{(r)}$ from period t to this period, while it suggests to them to choose action profile a^e otherwise. Since a^e is strictly dominant in G and $a^{(r)}$ is strictly dominant in $G^{[a^{(r)}]}$, it follows, from Theorem 1, that only the play according to s^* after period $T - \hat{t} + 1$ satisfies the perfect iterative undominance property in every subgame after period $T - \hat{t} + 1$.

Next, consider an arbitrary period $t \leq T - \hat{t}$ and an arbitrary history $h(t-1) \in H(t-1)$. Suppose that players will behave according to the strategy profile s^* after period $t+1$. We show below that in period t players will choose a strictly dominant action profile, which is the same as the one suggested by s^*.

Suppose that there exists no player who has deviated in previous periods. Then, the strategy profile s^* suggests to players to choose the efficient action profile $(1,1)$ in period t. Note, from the definition of the side payment contract m^*, that no players will be fined, whether or not they choose action profile $(1,1)$ in period t, which is the same as the one suggested by s^*. If a player does not choose action 1 and the other player chooses (does not choose) action 1 in period t, then players' choices during the last $\hat{t}$ periods will be changed from the repeated choices of action profile $a^{(0)}$ to the repeated choices of action profile $a^{(1)}$, which is less preferable than action profile $a^{(0)}$, (from the repeated choices of action profile $a^{(1)}$ to the repeated choices of action profile $a^{(2)}$, which is less preferable than action profile $a^{(1)}$, respectively). Moreover, if a player does not choose action 1 in period t, then players' choices from period $t+1$ to period $T-\hat{t}$ will be either unchanged, or changed from the repeated choices of action profile $(1,1)$ to the repeated choices of action profile $(0,0)$, which is less preferable than action profile $(1,1)$. Precisely, the difference in long-run payoffs for each player $i=1,2$ between the choice of action 1 and the choice of any action $a_i(t)$ other than action 1 is equal to

$$\begin{aligned} &u_i(1,a_j(t))-u_i(a_i(t),a_j(t))+(T-\hat{t}-t)\{u_i(1,1)-u_i(0,0)\} \\ &+\hat{t}\{u_i(a^{(0)})-u_i(a^{(1)})\} \text{ if } a_j(t)=1, \end{aligned}$$

and

$$u_i(1,a_j(t))-u_i(a_i(t),a_j(t))+\hat{t}\{u_i(a^{(1)})-u_i(a^{(2)})\} \text{ if } a_j(t)\neq 1,$$

where $j\neq i$. Note from inequality (7) that this value is more than or equal to

$$-1+a_i+\hat{t}\cdot\min[1-\frac{\varepsilon M}{2},\frac{\varepsilon M}{4}]\geq -1+\frac{\varepsilon M\hat{t}}{4}>0.$$

Hence, the choice of action 1, which is the same as the action suggested by s_i^*, is strictly dominant in period t.

Finally, suppose that a player has deviated in previous periods. Then, the strategy profile s^* suggests to players to choose action profile $(0,0)$ in period t. Note, from the definition of m^*, that no players will be fined, whether or not they choose action profile $(0,0)$ in period t, which is the same as the one suggested by s^*. Note also, from the definition of s^*, that players' choices in period t never influence their play after period $t+1$ that is suggested by s^*. Hence, one gets from the fact that action profile $(0,0)$ is strictly dominant in G, that the choice of action 0, which is the same as the action suggested by s_i^*, is strictly dominant in period t.

By applying the above arguments recursively from period $t=T-\hat{t}$ to period $t=1$, we have concluded, from the definition of perfect iterative undominance, that s^* is the unique perfect iterative undominated strategy profile in $\Gamma(T, m^*)$, and therefore, the efficient payoff vector $u(1,1) = (1,1)$ is uniquely and virtually implementable in terms of perfect iterative undominance, provided that the number of repetitions T is large enough.

Remark 1: The supposition that a^e is the unique strict Nash equilibrium action profile in G, is necessary to show that s^* is the unique perfect equilibrium in $\Gamma(T, m^*)$ provided that εM is so close to zero. Suppose that there exists another strict Nash equilibrium $\bar{a} \neq a^e$ in G. Then, for every $r \in \{0,...,n\}$, and every $\varepsilon M > 0$ sufficiently close to zero, not only $a^{(r)}$ but also $\bar{a}$ is a Nash equilibrium action profile in $G^{[a^{(r)}]}$. This implies that the strategy profile that always suggests to players to choose $\bar{a}$ in every period, is a perfect equilibrium in $\Gamma(T, m^*)$ as well as s^*.

Remark 2: Note that we cannot obtain the same result as Theorem 2 when we replace the side payment contract m^* with the side payment contract that will fine a player if and only if she is the last deviant from s^*, whether or not she was the last deviant in the period before period $T - \hat{t} + 1$. Suppose that, given a history $h(T - \hat{t}) \in H(T - \hat{t})$, player 1 is the single last deviant from s^* before period $T - \hat{t} + 1$, and suppose also that the other players behave according to s^* after period $T - \hat{t} + 1$. Then, the strategy profile s^* suggests to players to continually choose action profile $a^{(1)}$ after period $T - \hat{t} + 1$. Since player 1 will be (randomly) fined, whether or not she deviates from $a^{(1)}$, and since $a^{(1)}$ is not a Nash equilibrium in G, player 1 has an incentive to choose an action other than action $a_1^{(1)}$ in the final period T. This implies that s^* is not a perfect equilibrium in a finitely repeated game with this side payment contract.

5. Other Side Payment Contracts

In the previous sections, we have constructed the side payment contracts such that a player will be fined *only* if she is the last deviant from a target strategy profile. The device of stipulating that only the last deviants will be fined plays a crucial role in eliminating unwanted equilibria in the subgames with which players are confronted after someone has deviated.[9]

However, we do not need to use this device when a^e is either the unique Nash equilibrium action profile or the unique iteratively undominated action profile in G. For example, let a side payment contract m^{++} be specified, such that a player will be fined if and only if she is the *first* deviant from s^+. We can confirm, in the same way as in Theorem 1, that s^+ is the unique perfect equilibrium (the unique perfect iteratively undominated strategy profile) in $\Gamma(T, m^{++})$, if a^e is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in G and a^* is the unique Nash equilibrium action profile (the unique iteratively undominated action profile, respectively) in $G^{[a^*]}$.

In addition, let a side payment contract m^{**} be specified, such that a player will be fined if and only if she is the *first* deviant from the strategy profile s^* *during the last* $\hat{t}$ *periods*. We can confirm, in the same way as in Theorem 2, that s^* is the unique perfect equilibrium (the unique perfect iteratively undominated strategy profile) in $\Gamma(T, m^{**})$, if a^e is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in G and $a^{(r)}$ is the unique Nash equilibrium action profile (the unique iteratively undominated action profile, respectively) in $G^{[a^{(r)}]}$ for every $r \in \{0, ..., n-1\}$.

[9] This point was firstly raised by Glazer and Perry (1996) in their study of the implementation of social choice functions.

6. Conclusion

This paper investigated a finitely repeated game in which side payments among players are possible but the complete history of their action choices is hardly verifiable to the court. We showed that every payoff vector induced by an action profile that Pareto-dominates a Nash equilibrium action profile is uniquely and exactly implementable even if the probability of the complete history being verifiable is small and the amount of fines is negligible compared with the differences in long-run payoffs. Moreover, we showed, with minor restrictions on the component game, that this payoff vector is also uniquely and virtually implementable even if this probability is close to zero and the amount of fines is negligible compared with the differences not only in long-run payoffs but also in instantaneous payoffs.

We have the following problems to be solved in future research.

The present paper assumed that monitoring is perfect. In reality, however, players may only imperfectly monitor their opponents' action choices. In such cases, players observe random signals that are determined according to probability functions conditional on players' action choices, but these signals may be hardly verifiable to the court.[10] It is important, as a next step, to investigate finitely repeated games with side payments in which players can write a side payment contract which is contingent not on the complete history of their action choices but on the complete history of observed signals, on the assumption that the complete history of observed signals is not unverifiable but hardly verifiable to the court.

The present paper assumed that the side payment contract has full commitment power. In real situations, however, players may multilaterally breach and renegotiate on the terms of their explicit agreement in every period. Moreover, many real economic relationships are governed by a sequence of short-term contracts.[11] It is also important,

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[10] Several works such as Fudenberg, Levine and Maskin (1990) have considered infinitely repeated games with imperfect monitoring and provided the Folk Theorem or the efficiency result on the assumption that the signals are *publicly* observable among players but never verified to the court. Recent works such as Kandori and Matsushima (1998) and Matsushima (2000) investigated infinitely repeated games in which these signals are only *privately* observed by players, providing also their own respective Folk Theorems.

[11] Several works such as Fudenberg and Tirole (1990) and Hermalin and Katz (1991) investigated the renegotiation-proofness on the terms of an explicit agreement in the single-period single-agent problem. Fudenberg, Holmstrom and Milgrom (1990) and Rey and

as another next step, to investigate finitely repeated games with side payments in which players renegotiate not only on the terms of their implicit agreement but also on the terms of the explicit agreement, and in which explicit contracts that players agree to write are restricted to cover only a fraction of the whole duration of their relationship.

It is unquestionably important to also extend the present paper to finitely repeated games without strict Nash equilibrium action profiles, infinitely repeated games, repeated games with incomplete information, and general stochastic games. However, such unsolved problems exceed the purpose of this paper.

Salanie (1990) investigated the case of short-term commitment with renegotiation in the multi-period single-agent problem.

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Appendix

Proof of Theorem 1: The first part of this proof will consider the situation in which a^e and a^* are the unique Nash equilibrium action profiles in $G^{[a^e]}$ and in $G^{[a^*]}$, respectively. Fix $t = 1,...,T$ arbitrarily. Suppose that s^+ satisfies the perfect equilibrium property in every subgame after period $t+1$, i.e., s^+ satisfies inequalities (1) for all $t' = t,...,T-1$, and that for every perfect equilibrium $s \in S$ in $\Gamma(T, m^+)$,

$$s(h(t')) = s^+(h(t')) \text{ for all } t' = t,...,T-1 \text{ and all } h(t') \in H(t'). \tag{A-1}$$

Consider an arbitrary perfect equilibrium $s \in S$ in $\Gamma(T, m^+)$ and an arbitrary history $h(t-1) \in H(t-1)$.

Suppose that $a(\tau) = a^*$ for all $\tau \le t-1$. The definition of m^+ implies that for every $a \in A$, and every $i \in N$, if $a_i \ne a_i^*$, then

$$m_i^+(h(s,(h(t-1), a/a_i^*))) - m_i^+(h(s,(h(t-1), a))) = M,$$

which, together with the definition of s^+ and inequality $u(a^*) > u(a^e)$, implies that

$$v_i(h(s,(h(t-1), a/a_i^*))) + \varepsilon m_i^+(h(s,(h(t-1), a/a_i^*)))$$

$$- v_i(h(s,(h(t-1), a))) - \varepsilon m_i^+(h(s,(h(t-1), a)))$$

$$\ge u_i(a/a_i^*) - u_i(a) + \varepsilon M. \tag{A-2}$$

Since a^* is the unique Nash equilibrium action profile in $G^{[a^*]}$, it follows, from inequalities (A-2), that only the play of a^* in period t followed by the choices according to s^+ from period $t+1$ satisfies the perfect equilibrium property in the subgame started from period t, and therefore, it must hold that $s(h(t-1)) = a^*$. Note that s^+ satisfies inequalities (1) for $h(t-1)$, because players choose a^* in period t according to s^+.

Next, suppose that $a(\tau) \ne a^*$ for some $\tau \in \{1,...,t-1\}$. The definition of m^+ implies that for every $a \in A$, and every $i \in N$, if $a_i \ne a_i^e$, and $a_j \ne a_j^e$ for some $j \ne i$, then

$$m_i^+(h(s,(h(t-1), a/a_i^e))) - m_i^+(h(s,(h(t-1), a))) = M,$$

which, together with the definition of s^+, implies that

$$v_i(h(s,(h(t-1), a/a_i^e))) + \varepsilon m_i^+(h(s,(h(t-1), a/a_i^e)))$$

$$-v_i(h(s,(h(t-1),a)))-\varepsilon m_i^+(h(s,(h(t-1),a)))$$

$$=u_i(a/a_i^e)-u_i(a)+\varepsilon M\,. \qquad \text{(A-3)}$$

The definition of m^+ again implies that for every $a\in A$, and every $i\in N$, if $a_i\neq a_i^e$, and $a_j=a_j^e$ for all $j\neq i$, then

$$m_i^+(h(s,(h(t-1),a/a_i^e)))-m_i^+(h(s,(h(t-1),a)))\geq 0\,,$$

which, together with the definition of s^+, implies that

$$v_i(h(s,(h(t-1),a/a_i^e)))+\varepsilon m_i^+(h(s,(h(t-1),a/a_i^e)))$$

$$-v_i(h(s,(h(t-1),a)))-\varepsilon m_i^+(h(s,(h(t-1),a)))$$

$$\geq u_i(a^e)-u_i(a^e/a_i)\,. \qquad \text{(A-4)}$$

Since a^e is the unique Nash equilibrium action profile in $G^{[a^e]}$, it follows, from equalities (A-3) and inequalities (A-4), that only the play of a^e in period t followed by the choices according to s^+ from period $t+1$ satisfies the perfect equilibrium property in the subgame started from period t and, therefore, it must hold that $s(h(t-1))=a^e$. Note that s^+ satisfies inequalities (1) for $h(t-1)$, because players choose a^e in period t according to s^+.

Hence, we have proved that s^+ satisfies inequalities (1) for all $t'=t-1,...,T-1$ and all $h(t')\in H(t')$, and that for every perfect equilibrium $s\in S$ in $\Gamma(T,m^+)$,

$$s(h(t'))=s^+(h(t')) \text{ for all } t'=t-1,...,T-1 \text{ and all } h(t')\in H(t')\,. \qquad \text{(A-5)}$$

Since the above supposition trivially holds for $t=T$, it follows, by backward induction, that s^+ is the unique perfect equilibrium in $\Gamma(T,m^+)$.

The latter part of this proof will consider the situation in which a^e and a^* are the unique iteratively undominated action profiles in $G^{[a^e]}$ and in $G^{[a^*]}$, respectively. Let $W^*(T)\equiv S$. Fix $t=1,...,T$ arbitrarily. Suppose that s^+ satisfies the perfect iterative undominance property in every subgame after period $t+1$, i.e., $s^+\in W^*(t)$, and that equalities (A-1) hold for every $s\in W^*(t)$. We consider an arbitrary history $h(t-1)\in H(t-1)$ and an arbitrary strategy profile $s\in W^\infty(h(t-1))$.

Suppose that $a(\tau)=a^*$ for all $\tau\leq t-1$. Since a^* is the unique iteratively undominated action profile in $G^{[a^*]}$, it follows, from inequalities (A-2), that only the play of a^* in period t followed by the choices according to s^+ from period $t+1$

satisfies the perfect iterative undominance property in the subgame started from period t and, therefore, it must hold that $s(h(t-1)) = a^*$. Note $s^+ \in W^\infty(h(t-1))$, because players choose a^* in period t according to s^+.

Next, suppose that $a(\tau) \neq a^*$ for some $\tau \in \{1,...,t-1\}$. Since a^e is the unique iteratively undominated action profile in $G^{[a^e]}$, it follows, from equalities (A-3) and inequalities (A-4), that only the play of a^e in period t followed by the choices according to s^+ from period $t+1$ satisfies the perfect iterative undominance property in the subgame started from period t and, therefore, it must hold that $s(h(t-1)) = a^e$. Note $s^+ \in W^\infty(h(t-1))$, because players choose a^e in period t according to s^+.

Hence, we have proved that $s^+ \in W^*(t-1)$, and that equalities (A-5) hold for every $s \in W^*(t-1)$. Since the above supposition trivially holds for $t = T$, it follows, by backward induction, that s^+ is the unique perfect iteratively undominated strategy profile in $\Gamma(T, m^+)$.

Q.E.D.

Proof of Theorem 2: The first part of this proof will consider the situation in which $a^{(r)}$ is the unique Nash equilibrium action profile in $G^{[a^{(r)}]}$ for every $r \in \{0,...,n\}$. Fix $r \in \{0,\ldots,n\}$ arbitrarily. Consider an arbitrary history $h(T-\hat{t}) \in H(T-\hat{t}, r)$. Since $a^{(r)}$ and $a^{(0)}$ are the unique Nash equilibrium action profiles in $G^{[a^{(r)}]}$ and in $G^{[a^{(0)}]}$, respectively, it follows, in the same way as in Theorem 1, that only the play according to s^* satisfies the perfect equilibrium property in the subgame started from period $T-\hat{t}+1$ with $h(T-\hat{t})$.

Next, consider an arbitrary period $t \leq T-\hat{t}$. Suppose that s^* satisfies the perfect equilibrium property in every subgame after period $t+1$, i.e., s^* satisfies inequalities (1) for all $t' = t,...,T-1$ and all $h(t') \in H(t')$, and that for every perfect equilibrium $s \in S$ in $\Gamma(T, m^*)$,

$$s(h(t')) = s^*(h(t')) \text{ for all } t' = t,...,T-1 \text{ and all } h(t') \in H(t'). \tag{A-6}$$

Consider an arbitrary perfect equilibrium $s \in S$ in Γ and an arbitrary history $h(t-1) \in H(t-1)$. Note from the definition of m^* that for every $a \in A$, and every $i \in N$,

$$m_i^*(h(s,(h(t-1),a))) = 0.$$

Suppose that $a(\tau) = a^*$ for all $\tau \leq t-1$. The definitions of s^* and m^* imply that for every $a \in A$ and every $i \in N$, if $a_i \neq a_i^*$, then

$$v_i(h(s,(h(t-1),a/a_i^*))) + \varepsilon m_i^*(h(s,(h(t-1),a/a_i^*)))$$

$$-v_i(h(s,(h(t-1),a))) - \varepsilon m_i^*(h(s,(h(t-1),a)))$$

$$\geq u_i(a/a_i^*) - u_i(a) - \hat{t}\{u_i(a^{(n(a,a^*))}) - u_i(a^{(n(a,a^*)-1)})\}. \qquad \text{(A-7)}$$

Inequalities (A-7), together with inequalities (5), imply that only the play of a^* in period t followed by the choices according to s^* from period $t+1$ satisfies the perfect equilibrium property in the subgame from period t. Hence, it must hold that $s(h(t-1)) = a^*$. Note that s^* satisfies inequalities (1) for $h(t-1)$, because players choose a^* in period t according to s^*.

Next, suppose that $a(\tau) \neq a^*$ for some $\tau \in \{1,...,t-1\}$. The definitions of s^* and m^* imply that for every $a \in A$, and every $i \in N$, if $a_i \neq a_i^e$, then

$$v_i(h(s,(h(t-1),a/a_i^e))) + \varepsilon m_i^*(h(s,(h(t-1),a/a_i^e)))$$

$$-v_i(h(s,(h(t-1),a))) - \varepsilon m_i^*(h(s,(h(t-1),a)))$$

$$= u_i(a/a_i^e) - u_i(a). \qquad \text{(A-8)}$$

Since a^e is the unique Nash equilibrium action profile in G, it follows, from equalities (A-8), that only the play of a^e in period t followed by the choices according to s^* from period $t+1$ satisfies the perfect equilibrium property in the subgame started from period t. Hence, it must hold that $s(h(t-1)) = a^e$. Note that s^* satisfies inequalities (1) for $h(t-1)$, because players choose a^e in period t according to s^*.

Hence, we have proved that s^* satisfies inequalities (1) for all $t' = t-1,...,T-1$ and all $h(t') \in H(t')$, and that for every perfect equilibrium $s \in S$ in $\Gamma(T,m^*)$,

$$s(h(t')) = s^*(h(t')) \text{ for all } t' = t-1,...,T-1 \text{ and all } h(t') \in H(t'). \qquad \text{(A-9)}$$

Since the above supposition trivially holds for $t = T$, it follows, by backward induction, that s^* is the unique perfect equilibrium in $\Gamma(T,m^*)$.

The latter part of this proof will consider the situation in which $a^{(r)}$ is the unique iteratively undominated action profile in $G^{[a^{(r)}]}$ for every $r \in \{0,...,n\}$. Fix $r \in \{0,\dots,n\}$ arbitrarily. Consider an arbitrary history $h(T-\hat{t}) \in H(T-\hat{t},r)$. Since

$a^{(r)}$ and $a^{(0)}$ are the unique iteratively undominated action profiles in $G^{[a^{(r)}]}$ and in $G^{[a^{(0)}]}$, respectively, it follows, in the same way as in Theorem 1, that only the play according to the strategy profile s^* satisfies the perfect iterative undominance property in the subgame started from period $T-\hat{t}+1$ with $h(T-\hat{t})$.

Next, consider an arbitrary period $t \leq T-\hat{t}$. Suppose that s^* satisfies the perfect iterative undominance property in every subgame after period $t+1$, i.e., $s^* \in W^*(t)$, and that equalities (A-8) hold for every $s \in W^*(t)$. Consider an arbitrary history $h(t-1) \in H(t-1)$ and an arbitrary strategy profile $s \in W^\infty(h(t-1))$.

Suppose that $a(\tau)=a^*$ for all $\tau \leq t-1$. Inequalities (5) and (A-7) imply that only the play of a^* in period t followed by the choices according to s^* from period $t+1$ satisfies the perfect iterative undominance property in the subgame from period t. Hence, it must hold that $s(h(t-1))=a^*$. Note that $s \in W^\infty(h(t-1))$, because players choose a^* in period t according to s^*.

Next, suppose that $a(\tau) \neq a^*$ for some $\tau \in \{1,...,t-1\}$. Inequalities (A-8), together with the fact that a^e is the unique iteratively undominated action profile in G, imply that only the play of a^e in period t followed by the choices according to s^* from period $t+1$ satisfies the perfect iterative undominance property in the subgame started from period t. Hence, it must hold that $s(h(t-1))=a^e$. Note that $s \in W^\infty(h(t-1))$, because players choose a^e in period t according to s^*.

Hence, we have proved that $s^* \in W^*(t-1)$, and that equalities (A-6) hold for every $s \in W^*(t-1)$. Since the above supposition trivially holds for $t=T$, it follows, by backward induction, that s^* is the unique perfect iteratively undominated strategy profile in $\Gamma(T,m^*)$.

Q.E.D.

Proof of Proposition 3: Note from the definition of Δ that we can choose a positive real number $\eta>0$ close enough to zero to satisfy that $a^e+\eta\Delta$ belongs to A. The instantaneous payoff vector induced by the action profile $a^e+\eta\Delta$, i.e., $u(a^e+\eta\Delta)$, is approximated by

$$u(a^e)+\eta\sum_{i\in N}\left(\frac{\partial u_1(a^e)}{\partial a_i},...,\frac{\partial u_n(a^e)}{\partial a_i}\right)\Delta_i,$$

which is larger than $u(a^e)$ because of inequality (6). Hence, there exists an infinite sequence of action profiles $(a^m)_{m=1}^\infty$ satisfying that $\lim_{m\to\infty} a^m = a^e$ and $u(a^m)>u(a^{m+1})$ for all $m \geq 1$. The continuity of u_i implies that for every large enough m, a^m is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in $G^{[a^m]}$ if a^e is a strict Nash equilibrium action profile and the unique Nash equilibrium action profile (the unique perfect iteratively undominated action profile, respectively) in G. Hence, we can choose $(a^{(r)})_{r=0}^{n}$ satisfying inequalities (4) and satisfying that for every $r \in \{0,\ldots,n\}$, $a^{(r)}$ is the unique Nash equilibrium action profile (the unique iteratively undominated action profile) in $G^{[a^{(r)}]}$ if a^e is the unique Nash equilibrium action profile (the unique iteratively undominated action profile, respectively) in G.

Q.E.D.

	c	d	d'
c	1, 1	−3, 2	−3, −5
d	2, −3	0, 0	−3, −5
d'	−3, −5	−3, −5	2, −2

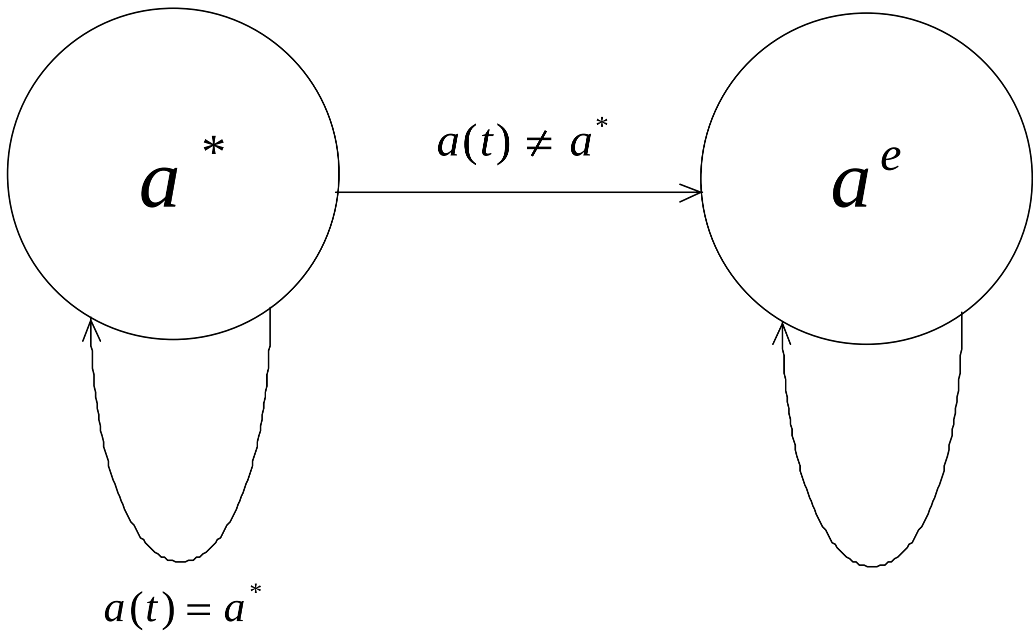
[Figure 1: Component Game *G*]

	c	d	d'
c	1, 1	−3, −4	−3, −11
d	−4, −3	−6, −6	−9, −11
d'	−9, −5	−9, −11	−4, −8

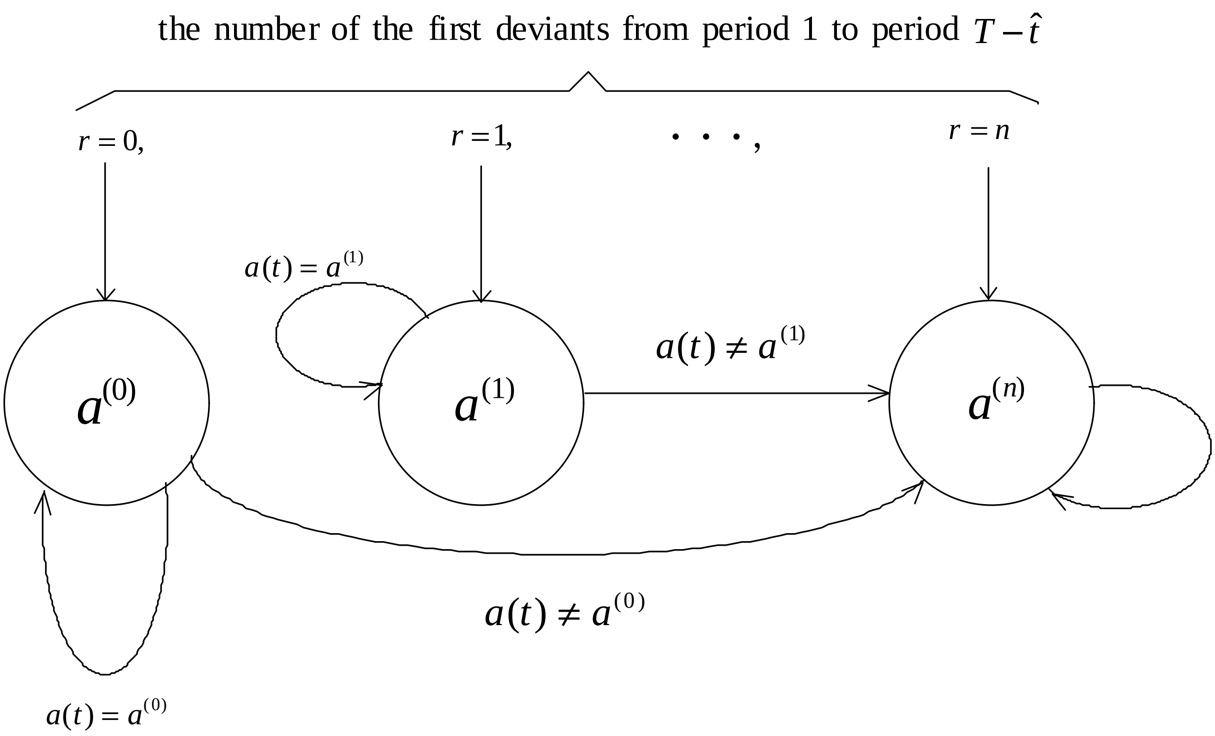
[Figure 2: Component Game $G^{[(c,c)]}$]

	c	d	d'
c	$-5,\ -5$	$-9,\ 2$	$-9,\ -11$
d	$2,\ -9$	$0,\ 0$	$-3,\ -11$
d'	$-9,\ -11$	$-9,\ -5$	$-4,\ -8$

[Figure 3: Component Game $G^{[(d,d)]}$]



[Figure 4: Modified Trigger Strategy Profile:
From Period 1 to Period $T - \hat{t}$]



[Figure 5: Modified Trigger Strategy Profile:

From Period $T-\hat{t}+1$ to Period T]