CIRJE-F-84

The Folk Theorem with Private Monitoring and Uniform Sustainability

Hitoshi Matsushima
University of Tokyo

July 2000

The Folk Theorem with Private Monitoring and Uniform Sustainability*

Hitoshi Matsushima+

University of Tokyo

The First Version: April 4, 2000
The Current Version: July 28, 2000

---

* The first version of this paper was entitled "Private Monitoring, Likelihood Ratio Conditions, and the Folk Theorem," which was presented at the Microeconomic Theory Conference held by the Cowles Foundation at Yale University on April 28 and 29, 2000. In the current version I have made a lot of substantial changes. For example, the first version only showed that the Folk Theorem holds under the zero likelihood ratio condition, whereas the current version shows the Folk Theorem *without* the zero likelihood ratio condition. I am grateful to Dilip Abreu for encouraging me to complete this work. I also thank Stefan Worrall very much for his careful reading. All errors are mine.

+ Faculty of Economics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan. E-mail: hitoshi@e.u-tokyo.ac.jp

Abstract

This paper investigates infinitely repeated prisoner-dilemma games where the discount factor is less than but close to 1. We assume that monitoring is truly imperfect and truly private, there exist no public signals and no public randomization devices, and players cannot communicate and use only pure strategies. We show that implicit collusion can be sustained by Nash equilibria under a mild condition. We show that *the Folk Theorem* holds when players' private signals are *conditionally independent*. These results are permissive, because we require no conditions concerning the accuracy of private signals such as the zero likelihood ratio condition. We also investigate the situation in which players play a Nash equilibrium of a machine game irrespective of their initial states, i.e., they play a *uniform equilibrium*. We show that there exists a unique payoff vector sustained by a uniform equilibrium, i.e., a *unique uniformly sustainable payoff vector*, which *Pareto-dominates* all other uniformly sustainable payoff vectors. We characterize this payoff vector by using the values of the minimum likelihood ratio. We show that this payoff vector is efficient *if and only if* the zero likelihood ratio condition is satisfied. These positive results hold even if each player has *limited knowledge* on her opponent's private signal structure.

Keywords: Repeated Prisoner-Dilemma Games, Private Monitoring, Conditional Independence, Folk Theorem, Uniform Sustainability, Zero Likelihood Ratio Condition, Limited Knowledge.

1. Introduction

This paper investigates infinitely repeated prisoner-dilemma games where the discount factor is less than but close to 1. We assume that players not only imperfectly but also *privately* monitor their opponents' actions. Players cannot observe their opponents' actions directly, but can only observe their own private signals which are drawn according to a density function over closed intervals conditional upon the action profile played. There are no public signals.

The paper investigates the possibility that implicit collusion can be sustained by Nash equilibria. We show that there exist Nash equilibrium payoff vectors which are better than the one-shot Nash equilibrium payoff vector if the minimum of the likelihood ratio indicating whether the opponent has chosen the right action satisfies a mild condition. Furthermore, we show that an *efficient* payoff vector is approximated by a Nash equilibrium payoff vector if this minimum likelihood ratio is equal to *zero*, i.e., if for each player there exists a private signal that indicates accurately whether her opponent has chosen the right action. Note that as this signal is *not* public this efficiency result is not immediate.

Given the zero likelihood ratio condition, it is well known that efficiency is attainable in the limit of the discount factor even with imperfect monitoring provided that monitoring is public. With private monitoring the problem is more delicate. Even when a player is certain that a particular opponent has deviated, this certainty will typically not be shared by the other players and they will be unable to coordinate on an equilibrium which punishes the deviant in the continuation game. Nevertheless a more complicated argument establishes that efficiency is attainable under this condition.

We also intensively investigate the situation in which players' private signals are *conditionally independent*, i.e., players can obtain *no* information on what their opponents have observed by observing their own private signals. We show, as the main theorem of this paper, that the *Folk Theorem* holds, i.e., every feasible and individually rational payoff vector is approximated by a Nash equilibrium payoff vector, provided that players' private signals are conditionally independent. This result is permissive, because we require *no* informational conditions concerning the accuracy of players' private signals such as the zero likelihood ratio condition.

The study of repeated games with private monitoring is relatively new. Most earlier work in this area has assumed that monitoring is either perfect or public and has investigated only perfect public equilibria. Perfect public equilibrium requires that the past histories relevant to future play are common knowledge in every period. This common knowledge property makes equilibrium analyses tractable, because players' future play can always be described as a Nash equilibrium. When monitoring is only private, however, it is inevitable that an equilibrium sustaining implicit collusion depends on players' private histories, and therefore, the past histories relevant to future play are not common knowledge. This makes equilibrium analyses much more difficult, especially in the discounting case, because players' future play cannot be described as a Nash equilibrium.

To the best of my knowledge, Radner (1986) is the first paper on repeated games with private monitoring. Radner assumed no discounting, and showed that every feasible and individually rational payoff vector can be sustained by a Nash equilibrium.[1] The two papers by Matsushima (1990a, 1990b) appear to be the first to investigate the discounting case. Matsushima (1990a) provided an Anti-Folk Theorem, showing that it is impossible to sustain implicit collusion by pure strategy Nash equilibria when private signals are conditionally independent and Nash equilibria are restricted to be independent of payoff-irrelevant private histories. The present paper establishes the converse result: the Folk Theorem *holds* when we use pure strategy Nash equilibria which can depend on payoff-irrelevant private histories.

Matsushima (1990b) conjectured that a Folk Theorem type result could be obtained even with private monitoring and discounting when players can communicate by making publicly observable announcements. Subsequently, Kandori and Matsushima (1998) and Compte (1998) proved the Folk Theorem with communication. Communication synthetically generates public signals and consequently it is possible to conduct the dynamic analysis in terms of perfect public equilibria as in the paper by Fudenberg, Levine and Maskin (1994) on the Folk Theorem with imperfect public monitoring. The present paper assumes that players make *no* publicly observable announcements.

Interest in repeated games with private monitoring and no communication has been stimulated by a number of recent papers, including Sekiguchi (1997), Bhaskar (1999), Piccione (1998), and Ely and Valimaki (1999). Sekiguchi (1997) investigated a restricted class of prisoner-dilemma games on the assumption that monitoring was *almost perfect* and that players' private signals were conditionally independent. Sekiguchi was the first to show that an efficient payoff vector can be approximated by a mixed strategy Nash equilibrium payoff vector even if players cannot communicate. By

[1] See also Lehler (1989) and Fudenberg and Levine (1991) for the study of repeated games with no discounting and with private monitoring.

using public randomization devices, Bhaskar (1999) extended Sekiguchi's result to more general prisoner-dilemma games.

Piccione (1998) and Ely and Valimaki (1999) also considered repeated prisoner-dilemma games when the discount factor is close to 1, and provided their respective Folk Theorems. Both papers constructed mixed strategy equilibria in which each player is indifferent between the right action and the wrong action irrespective of her opponent's possible future strategy. Piccione used dynamic programming techniques over infinite state spaces, while Ely and Valimaki used two-state Markov strategies. Both papers investigated only the almost-perfect monitoring case, and most of their arguments rely heavily on this assumption. However, in the last section of his paper, Piccione provides an example in which implicit collusion is possible even if players' private observation errors are not infinitesimal.

Mailath and Morris (1998) investigate the robustness of perfect public equilibria when monitoring is almost public, i.e., each player can always discern accurately which private signal her opponent has observed by observing her own private signal. The present paper does *not* assume that monitoring is almost public.

In consequence, this paper has many substantial points of departure from the earlier literature. We assume that there exist no public signals, players make no publicly observed announcements, and there exist no public randomization devices. We do not require that monitoring is either almost perfect or almost public. Hence, the present paper can be regarded as the first work to provide affirmative answers to the possibility of implicit collusion with discounting when monitoring is *truly imperfect* and *truly private*.

As such, this paper may offer important economic implications within the field of industrial organization. In the real economy, communication between rival firms' executives is restricted by Anti-Trust Law, on the assumption that such communication enhances the possibility of a self-enforcing *cartel* agreement.[2] Moreover, in reality, firms usually cannot directly observe the prices or quantities of rival firms and the aggregate level of consumer demand is stochastic. Instead, each firm's only information about its opponents' actions within any particular period, is its own realized sales level and, therefore, each firm cannot know what its opponents have observed. These

---

[2] See such industrial organization textbooks as Scherer and Ross (1990) and Tirole (1988). Matsushima (1990b), Kandori and Matsushima (1998) and Compte (1998) provided a justification of why communication is so important for the self-enforcement of a cartel agreement.

circumstances tend to promote the occurrence of *price wars*, as each firm cannot know whether a fall in its own sales is due to a fall in demand or a *secret price cut* by a rival firm. In this way, it has been widely believed that a cartel agreement is most likely to be breached when each firm's monitoring of its opponents' actions is *truly* private.[3] In contrast, the present paper shows that collusive behavior is possible even if communication is prohibited and each firm obtains no public information on the prices or quantities of its rivals.

This paper is closely related to Piccione (1998) and Ely and Valimaki (1999), particularly the latter. This paper is also related to Matsushima (1999), which investigated the impact of multimarket contact on implicit collusion in the imperfect public monitoring case and provided the efficiency result by using the idea of a *review strategy* equilibrium. Our equilibrium construction may be viewed as extending the equilibrium construction of Ely and Valimaki combined with that of Matsushima to general private signal structures.

The latter part of this paper, i.e., Sections 7 and 8, are devoted to considering situations in which players have *limited knowledge* on their opponents' strategies. Both sections provide their respective sets of multiple possible strategies for each player. Each player only knows that her opponent plays one of these possible strategies, but has no idea which strategy is the correct one. We assume that it is common knowledge that the played strategy profile satisfies the Nash equilibrium property, while which Nash equilibrium is the correct one is not common knowledge.[4] The purpose of these sections is to clarify the possibility of implicit collusion even when players have limited knowledge on their opponents' strategies.

Section 7 regards a repeated prisoner-dilemma game as a *machine game* as explored by, for instance, Rubinstein (1984), Neyman (1985), and Abreu and Rubinstein (1987). A player behaves according to a machine which is defined as a combination of an output function, a transition function, and an initial state of machine. A rule for player

---

[3] Stigler (1964) is closely related. Moreover, Green and Porter (1984) investigated repeated quantity-setting oligopoly when the market demand is stochastic and firms cannot observe the quantities of their rival firms. They assumed that firms can publicly observe the market-clearing price. In contrast, the present paper assumes that there exist *no* publicly observable signals such as the market-clearing price.

[4] In the analysis of situations in which strategies are not common knowledge, Bernheim (1984) and Pearce (1984) introduced the concept of rationalizability instead of assuming that it is common knowledge that players' behaviors are described as a Nash equilibrium.

i is defined as a combination of an output function and a transition function. We assume that players' rules are common knowledge, but their initial states are not common knowledge. Each player knows that her opponent's play is consistent with this rule, but has *no* idea which initial state is the correct one. A rule profile is called a *uniform equilibrium* if every machine (i.e., rule plus initial state) profile consistent with this rule profile is a Nash equilibrium. Hence, all possible Nash equilibria are *interchangeable*. A payoff vector is called *uniformly sustainable* if there exists a uniform equilibrium such that every machine profile consistent with it induces virtually the same payoff vector as the given payoff vector. Hence, all possible Nash equilibria virtually induce this given payoff vector, i.e., are *virtually payoff-equivalent*.

We show that there exists a *unique* uniformly sustainable payoff vector which *Pareto-dominates* all other uniformly sustainable payoff vectors. This Pareto-dominance property is in sharp contrast with the fact that there exist a continuum/countable set of Pareto-undominated perfect equilibrium payoff vectors. We characterize this Pareto-dominant uniformly sustainable payoff vector by using the values of the minimum likelihood ratio. We show that this payoff vector is efficient *if and only if* the zero likelihood ratio condition is satisfied. Hence, the zero likelihood ratio condition is not only sufficient but also necessary for efficient uniform sustainability.

Abreu, Pearce and Stacchetti (1986) is related to this analysis. They investigated symmetric repeated oligopoly with imperfect public monitoring modeled by Green and Porter (1984), and characterized the optimal symmetric equilibrium, where the future punishment is triggered by the observation of the public signals which correspond to the minimum likelihood ratio. This optimal symmetric equilibrium is efficient if and only if the minimum likelihood ratio is equal to zero. In contrast with their work, the present paper does not assume that the model is symmetric, equilibria are restricted to be symmetric, or that there exists any public signal.

Ely and Valimaki (1999) is also, and more closely, related. In their analysis of Prisoner-Dilemma games with almost-perfect monitoring, Ely and Valimaki constructed interchangeable Markov strategy Nash equilibria. A point of difference is that Ely and Valimaki did not require that Nash equilibria are virtually payoff-equivalent, whereas the present paper does.

Section 8 considers the situation in which players have limited knowledge on their private signal structure. Each player knows her own private signal structure, i.e., knows the conditional density function of her own private signal, but does *not* know her opponent's private signal structure, i.e., does not know the conditional density function of her opponent's private signal. Each player's strategy depends on her own private signal structure, but is independent of her opponent's private signal structure.

We provide the following two positive results. We reconsider the sustainability of Nash equilibria and clarify whether the Folk Theorem can be achieved by using only players' strategies which depend only on their own private signal structures. Each player behaves according to a mapping which assigns a strategy for this player to each possible conditional density function over her own private signal. Their mappings are assumed to be common knowledge, but each player does not know which strategy in the range of the opponent's mapping is actually played. We require that every pair of strategies in the ranges of their mappings are Nash equilibria. We establish the *Folk Theorem* with *interchangeability* and *virtual payoff-equivalence*. That is, if it is common knowledge among players that private signals are conditionally independent, then, for every feasible and individually rational payoff vector, there exists a profile of mappings assigning each possible private signal structure a Nash equilibrium which induces approximately the same payoff vector as this payoff vector. Hence, all possible Nash equilibria can be regarded as being interchangeable and virtually payoff-equivalent.

We also reconsider uniform sustainability discussed in Section 7 and show that the Pareto-dominant uniformly sustainable payoff vector can be uniformly sustained by using *only* players' rules which depends only on their own private signal structures. Each player behaves according to a mapping which assigns a rule for this player to each possible conditional density function over her own private signal. Their mappings are assumed to be common knowledge, but each player does not know which rule in the range of the opponent's mapping is the correct one. We require that every pair of rules in the ranges of their mappings are uniform equilibria. Hence, all possible uniform equilibria can be regarded as being interchangeable. We do not require the conditional independence assumption. We show that the arguments in Section 7 hold even if each player only knows her own private signal structure, i.e., we show that there exists a profile of mappings which assigns each possible private signal structure a uniform equilibrium such that every machine profile consistent with it induces virtually the same payoff vector as the associated Pareto-dominant payoff vector.

The organization of this paper is as follows. Section 2 defines the model. Section 3 provides a theorem which characterizes a subset of sustainable payoff vectors. Section 4 gives the proof of this theorem. Section 3 shows that efficiency is sustainable under the zero likelihood ratio condition. Section 5 assumes conditional independence and provides the Folk Theorem. Section 6 gives the proof of this Folk Theorem. Section 7 shows that there exists the Pareto-dominant uniformly sustainable payoff vector, and this payoff vector is efficient if and only if the zero likelihood ratio condition is satisfied. Section 8 considers two scenarios in which each player has limited knowledge on her opponent's private signal structure, and shows that the positive results provided in the previous sections hold in each of these scenarios. Section 9 concludes.

2. The Model

An infinitely repeated prisoner-dilemma game $\Gamma(\delta) = ((A_i, u_i, \Omega_i)_{i=1,2}, \delta, p)$ is defined as follows. In every period $t \geq 1$, players 1 and 2 play a prisoner-dilemma game $(A_i, u_i)_{i=1,2}$. Throughout this paper, we will denote $j \neq i$, i.e., denote $j = 1$ when $i = 2$, and $j = 2$ when $i = 1$. Player $i's$ set of actions is given by $A_i = \{c_i, d_i\}$. Let $A \equiv A_1 \times A_2$. Player $i's$ instantaneous payoff function is given by $u_i: A \to R$. We assume that for every $i = 1,2$, $u_i(c) = 1$, $u_i(d) = 0$, $u_i(d / c_j) = 1 + x_i > 1$, and $u_i(c / d_j) = -y_i < 0$, where we denote $c \equiv (c_1, c_1)$ and $d \equiv (d_1, d_2)$. We assume also that $x_1 + x_2 \leq y_1 + y_2$, i.e., the payoff vector $(1,1)$ is *efficient*. The *feasible* set of payoff vectors $V \subset R^2$ is defined as the convex hull of the set $\{(1,1), (0,0), (1 + x_1, -y_2), (-y_1, 1 + x_2)\}$. The discount factor is denoted by $\delta \in [0,1)$. At the end of every period, each player i observes her own *private* signal ω_i. The set of player $i's$ private signals is defined as $\Omega_i \equiv [0,1]$. Let $\Omega \equiv \Omega_1 \times \Omega_2$. A signal profile $\omega \equiv (\omega_1, \omega_2) \in \Omega$ is determined according to a conditional density function $p(\omega | a)$. Let $p_i(\omega_i | a) \equiv \int_{\omega_j \in \Omega_j} p(\omega | a) d\omega_j$. We assume that $p_i(\omega_i | a)$ is continuous w. r. t. $\omega_i \in \Omega_i$, $p_i(\omega_i | a) > 0$ for all $a \in A$ and almost all $\omega_i \in \Omega_i$, and $p_i(\cdot | a) \neq p_i(\cdot | a')$ for all $a \in A$ and all $a' \in A / \{a\}$. Based on the above definitions, we may regard $u_i(a)$ as the expected value defined by

$$u_i(a) \equiv \int_{\omega_i \in \Omega_i} \pi_i(\omega_i, a_i) p_i(\omega_i | a) d\omega_i ,$$

where $\pi_i(\omega_i, a_i)$ is the realized instantaneous payoff for player i when player i chooses action a_i and observes her own private signal ω_i.

Remark: An example is the model of a *price-setting duopoly*. Actions c_i and d_i are regarded as the choices of *high* price $\lambda_i(c_i)$ and *low* price $\lambda_i(d_i)$, respectively, for firm $i's$ commodity, where $\lambda_i(c_i) > \lambda_i(d_i) \geq 0$. Firm $i's$ sales when private signal ω_i is observed is given by $q_i(\omega_i) \geq 0$. The realized instantaneous profit for firm i is given by $\pi_i(\omega_i, a_i) = \lambda_i(a_i) q_i(\omega_i) - C_i(q_i(\omega_i))$, where $C_i(q_i) \geq 0$ is firm $i's$ total cost of production.

A private history for player i up to period $t \geq 1$ is denoted by $h_i^t = (a_i(\tau), \omega_i(\tau))_{\tau=1}^t$, where $a_i(\tau) \in A_i$ is the action chosen by player i and $\omega_i(\tau) \in \Omega_i$ is the private signal observed by player i in period τ. The null history for player i is denoted by h_i^0. The set of all private histories for player i is denoted by

H_i. A (pure) *strategy for player* i is defined as a function $s_i : H_i \to A_i$. The set of strategies for player i is denoted by S_i. Let $S \equiv S_1 \times S_2$. Player $i's$ normalized long-run payoff induced by a strategy profile $s \in S$ is given by $v_i(\delta, s) \equiv (1-\delta)E[\sum_{t=1}^{\infty} \delta^{t-1} u_i(a(t)) | s]$. Let $v(\delta, s) \equiv (v_1(\delta, s), v_2(\delta, s))$. A strategy profile $s \in S$ is said to be a *Nash equilibrium* in $\Gamma(\delta)$ if for each $i = 1,2$ and every $s_i' \in S_i$, $v_i(\delta, s) \geq v_i(\delta, s / s_i')$. Since each player's private signal structure has almost full support, the set of Nash equilibrium payoff vectors is equivalent to the set of sequential equilibrium payoff vectors.

Definition 1: A payoff vector $v = (v_1, v_2) \in R^2$ is *sustainable* if for every $\varepsilon > 0$ and every infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ satisfying $\lim_{m \to +\infty} \delta^m = 1$, there exists an infinite sequence of strategy profiles $(s^m)_{m=1}^{\infty}$ such that for every large enough $m = 1,2,\ldots$, s^m is a Nash equilibrium in $\Gamma(\delta^m)$, and

$$v - (\varepsilon, \varepsilon) \leq \lim_{m \to +\infty} v(\delta^m, s^m) \leq v + (\varepsilon, \varepsilon).$$

Note that the set of sustainable payoff vectors is compact. We denote by $s_i|_{h_i^t}$ the strategy for player i induced by s_i after the private history $h_i^t \in H_i$ occurs.

3. Efficiency

The *likelihood ratio function* for player $i's$ private signals, $L_i : \Omega_i \times A^2 \to R$, is defined by

$$L_i(\omega_i, a, a') \equiv \begin{cases} \dfrac{p_i(\omega_i|a)}{p_i(\omega_i|a')} & \text{if } p_i(\omega_i|a') \neq 0 \\ \lim\limits_{\omega_i' \to \omega_i} \dfrac{p_i(\omega_i'|a)}{p_i(\omega_i'|a')} & \text{if } p_i(\omega_i|a') = 0 \end{cases}.$$

We assume that such a function L_i exists and is continuous w. r. t. $\omega_i \in \Omega_i$. We define the *minimum likelihood ratio function* for player i, $\underline{L}_i : A^2 \to R$, by

$$\underline{L}_i(a, a') \equiv \min_{\omega_i \in \Omega_i} L_i(\omega_i, a, a').$$

We define

$$\bar{v}_i \equiv 1 - \frac{\underline{L}_j(c, c/d_i) x_i}{1 - \underline{L}_j(c, c/d_i)},$$

and

$$\underline{v}_i \equiv \frac{\underline{L}_j(d, d/c_i) y_i}{1 - \underline{L}_j(d, d/c_i)}.$$

Let $\bar{v} \equiv (\bar{v}_1, \bar{v}_2)$ and $\underline{v} \equiv (\underline{v}_1, \underline{v}_2)$. Note that if for each $i = 1,2$,

$$1 > \frac{\underline{L}_j(c, c/d_i) x_i}{1 - \underline{L}_j(c, c/d_i)} + \frac{\underline{L}_j(d, d/c_i) y_i}{1 - \underline{L}_j(d, d/c_i)}, \quad (1)$$

then it holds that $\bar{v} > \underline{v}$. We define a subset $V^* \subset V$ by the convex hull of the set $\{(0,0), \bar{v}, (\bar{v}_1, \underline{v}_2), (\underline{v}_1, \bar{v}_2)\}$. See Figure 1.

[Figure 1]

Theorem 1: *If inequalities (1) hold, then every* $v \in V^*$ *is sustainable.*

We provide the proof of Theorem 1 in the next section.

Theorem 2: *If for each* $i = 1,2$,

$$\underline{L}_i(c, c/d_j) = 0, \quad (2)$$

and

$$\underline{L}_i(d, d/c_j) < \frac{1}{1 + y_j}, \quad (3)$$

then, $(1,1)$ *is sustainable.*

Proof: Equalities (2) and inequalities (3) imply inequalities (1). Equalities (2) implies $\bar{v} = (1,1)$. Hence, Theorem 1 implies that $(1,1)$ is sustainable.

Q.E.D.

Theorem 2 states that *the efficient payoff vector (1,1) can be approximately sustained by a Nash equilibrium when the minimum likelihood ratio* $\underline{L}_i(c, c/d_j)$ *between* c *and* c/d_j *is zero and the minimum likelihood ratio* $\underline{L}_i(d, d/c_i)$ *between* d *and* d/c_j *is less than a positive value* $\frac{1}{1+y_j}$ *for each* $i = 1,2$. Note that Theorems 1 and 2 do not depend on any informational assumption such as conditional independence.

4. Proof of Theorem 1

The proof of Theorem 1 is divided into three steps.[5]

Step 1: We show that for every $v^+ \in V^*$ and every $v^- \in V^*$, if

$$\bar{v} \geq v^+ > v^- \geq \underline{v},$$

then, v^+, v^-, (v_1^+, v_2^-) and (v_1^-, v_2^+) are all sustainable.

From the continuity of L_i, we can choose $\hat{w}_i \in \Omega_i$ for each $i = 1,2$ which satisfies

$$v_j^+ = 1 - \frac{L_i(\hat{w}_i, c, c/d_j)x_j}{1 - L_i(\hat{w}_i, c, c/d_j)},$$

that is,

$$L_i(\hat{w}_i, c, c/d_j) = \frac{1 - v_j^+}{x_j + 1 - v_j^+}. \tag{4}$$

For each $i = 1,2$, choose $x_j > 0$ close to 0. From the continuity of L_i, we can choose $\tilde{w}_i = \tilde{w}_i(x_j) \in \Omega_i$ for each $i = 1,2$ which satisfies

$$\frac{v_j^- + x_j}{y_j + v_j^- + x_j} > L_i(\tilde{w}_i, d, d/c_j) > \frac{v_j^-}{y_j + v_j^-}. \tag{5}$$

Let $e_i > 0$ and $\tilde{l}_i > 0$ be positive real numbers which are close to 0.

Consider the following Markov strategies with two states, i.e., "play c_i", and "play d_i". When player $i's$ state is "play c_i" and player i observes a private signal which belongs to (does not belong to) the interval $(\hat{w}_i - e_i, \hat{w}_i + e_i]$ in the current period, player $i's$ state in the next period will be "play d_i" ("play c_i", respectively). When player $i's$ state is "play d_i" and player i observes a private signal which belongs to (does not belong to) the interval $(\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]$ in the current period, player $i's$ state will be "play c_i" ("play d_i", respectively). See Figure 2.

[Figure 2]

According to Ely and Valimaki (1999), we require that each player $i = 1,2$ is indifferent between the choice of action c_i and the choice of action d_i *irrespective of her own private history*. This incentive constraint is much stronger than sequential equilibrium but

---

[5] This proof does not depend on the assumption of $x_1 + x_2 \leq y_1 + y_2$.

drastically simplifies equilibrium analyses. In the following proof, we show that for every discount factor close to 1, there exist e_i and $\tilde{l}_i$ for each $i = 1,2$ such that all of the four Markov strategy profiles associated with different initial state profiles satisfy this incentive constraint, i.e., are Nash equilibria, and virtually induce the payoff vectors v^+, v^-, (v_1^+, v_2^-) and (v_1^-, v_2^+).

Fix $i = 1,2$ arbitrarily. Choose $e_i > 0$ close to 0. From equality (4), we can choose $\hat{v}_j = \hat{v}_j(e_i)$ which satisfies

$$\frac{\int\limits_{w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]} p_i(w_i|c) dw_i}{\int\limits_{w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]} p_i(w_i|c/d_j) dw_i} = \frac{1 - \hat{v}_j}{x_j + 1 - \hat{v}_j}. \tag{6}$$

Note that $\hat{v}_j = \hat{v}_j(e_i)$ tends towards v_j^+ as e_i approaches 0. We define

$$a_j = a_j(e_i) \equiv \frac{\int\limits_{w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]} p_i(w_i|c/d_j) dw_i}{x_j + 1 - \hat{v}_j}. \tag{7}$$

Note that $a_j(e_i)$ tends towards 0 as e_i approaches 0.

We define $\overline{l}_i = \overline{l}_i(e_i)$ and $\underline{l}_i = \underline{l}_i(e_i)$ by

$$\overline{l}_i > \underline{l}_i > 0,$$

$$\int\limits_{w_i \in (\tilde{w}_i - \overline{l}_i, \tilde{w}_i + \overline{l}_i]} p_i(w_i|d/c_j) dw_i = (y_j + v_j^- + x_j) a_j,$$

and

$$\int\limits_{w_i \in (\tilde{w}_i - \underline{l}_i, \tilde{w}_i + \underline{l}_i]} p_i(w_i|d/c_j) dw_i = (y_j + v_j^-) a_j.$$

Note that both $\overline{l}_i(e_i)$ and $\underline{l}_i(e_i)$ tend towards 0 as e_i approaches 0, because $a_j = a_j(e_i)$ tends towards 0 as e_i approaches 0. Choose any continuous function $w_j = w_j(e_i, x_j) : [\underline{l}_i, \overline{l}_i] \to [v_j^-, v_j^- + x_j]$ which satisfies that

$$w_j(\overline{l}_i) = v_j^- + x_j,$$

$$w_j(\underline{l}_i) = v_j^-,$$

and for every $l_i \in [\underline{l}_i, \overline{l}_i]$,

$$\int\limits_{w_i \in (\tilde{w}_i - l_i, \tilde{w}_i + l_i]} p_i(w_i|d/c_j) dw_i = (y_j + w_j(l_i)) a_j. \tag{8}$$

Since $L_i(\tilde{w}_i, d, d/c_j)$ is approximated by

$$\frac{\int\limits_{w_i \in (\tilde{w}_i - \overline{l}_i, \tilde{w}_i + \overline{l}_i]} p_i(w_i|d) dw_i}{\int\limits_{w_i \in (\tilde{w}_i - \overline{l}_i, \tilde{w}_i + \overline{l}_i]} p_i(w_i|d/c_j) dw_i},$$

one gets from inequalities (5) that

$$\frac{w_j(\overline{l}_i)}{y_j + w_j(\overline{l}_i)} = \frac{v_j^- + x_j}{y_j + v_j^- + x_j} > \frac{\int_{w_i \in (\tilde{w}_i - \overline{l}_i, \tilde{w}_i + \overline{l}_i]} p_i(w_i | d) dw_i}{\int_{w_i \in (\tilde{w}_i - \overline{l}_i, \tilde{w}_i + \overline{l}_i]} p_i(w_i | d / c_j) dw_i}$$

$$> \frac{v_j^-}{y_j + v_j^-} = \frac{w_j(\underline{l}_i)}{y_j + w_j(\underline{l}_i)}.$$

Hence, the continuity of $w_j(l_i)$ implies that there exists $\tilde{l}_i = \tilde{l}_i(e_i, x_j)$ such that

$$\frac{\int_{w_i \in (\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]} p_i(w_i | d) dw_i}{\int_{w_i \in (\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]} p_i(w_i | d / c_j) dw_i} = \frac{w_j(\tilde{l}_i)}{y_j + w_j(\tilde{l}_i)}. \tag{9}$$

We define

$$\tilde{v}_j = \tilde{v}_j(e_i, x_j) \equiv w_j(e_i, x_j)(\tilde{l}_i(e_i, x_j)) = w_j(\tilde{l}_i(e_i, x_j)).$$

Note that $\tilde{v}_j = \tilde{v}_j(e_i, x_j)$ tends towards v_j^- as x_j approaches 0, because $\tilde{l}_i = \tilde{l}_i(e_i, x_j)$ tends towards 0 as x_j approaches 0. We define $d_j = d_j(e_i, x_j) \in (0,1)$ by

$$\frac{1 - d_j}{d_j} = (\hat{v}_j - \tilde{v}_j) a_j. \tag{10}$$

Note that d_j tends towards 1 as e_i approaches 0, because $a_j(e_i)$ tends towards 0 as e_i approaches 0.

Fix an infinite sequence of discount factors $(d^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m \to +\infty} d^m = 1$. The above arguments imply that there exists $(e_1^m, x_1^m, e_2^m, x_2^m)_{m=1}^{\infty}$ such that

$$\lim_{m \to \infty} (e_1^m, x_1^m, e_2^m, x_2^m) = (0,0,0,0),$$

and for every large enough m,

$$d^m = d_1(e_2^m, x_1^m) = d_2(e_1^m, x_2^m).$$

Choose $(\tilde{w}_i^m, \tilde{l}_i^m, \hat{v}_i^m, \tilde{v}_i^m)_{m=1}^{\infty}$ satisfying that for every large enough m, $\tilde{w}_i^m \equiv \tilde{w}_i(x_j^m)$, $\tilde{l}_i^m \equiv \tilde{l}_i(e_i^m, x_j^m)$, $\hat{v}_i^m \equiv \hat{v}_i(e_j^m)$, and $\tilde{v}_i^m \equiv \tilde{v}_i(e_j^m, x_i^m)$. From equalities (6), (7) and (10), one gets

$$\int_{w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]} p_j(w_j | c) dw_j = \left(\frac{1 - d^m}{d^m}\right)\left(\frac{1 - \hat{v}_i^m}{\hat{v}_i^m - \tilde{v}_i^m}\right),$$

and, therefore,

$$\hat{v}_i^m = 1 - d^m + d^m \{\hat{v}_i^m (1 - \int_{w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]} p_j(w_j | c) dw_j)$$

$$+ \tilde{v}_i^m \int_{w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]} p_j(w_j | c) dw_j \}. \tag{11}$$

From equalities (7) and (10), one gets

$$\int_{w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]} p_j(w_j | c/d_i) dw_j = (\frac{1-d^m}{d^m})(\frac{x_i + 1 - \hat{v}_i^m}{\hat{v}_i^m - \tilde{v}_i^m}),$$

and, therefore,

$$\hat{v}_i^m = (1-d^m)(1+x_i) + d^m\{\hat{v}_i^m(1 - \int_{w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]} p_j(w_j | c/d_i) dw_j)$$

$$+ \tilde{v}_i^m \int_{w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]} p_j(w_j | c/d_i) dw_j\}. \quad (12)$$

From equalities (8) and (10), one gets

$$\int_{w_j \in (\tilde{w}_j^m - \tilde{l}_j^m, \tilde{w}_j^m + \tilde{l}_j^m]} p_j(w_j | d/c_i) dw_j = (\frac{1-d^m}{d^m})(\frac{y_i + \tilde{v}_i^m}{\hat{v}_i^m - \tilde{v}_i^m}),$$

and, therefore,

$$\tilde{v}_i^m = (1-d^m)(-y_i) + d^m\{\hat{v}_i^m(1 - \int_{w_j \in (\tilde{w}_j^m - \tilde{l}_j^m, \tilde{w}_j^m + \tilde{l}_j^m]} p_j(w_j | d/c_i) dw_j)$$

$$+ \tilde{v}_i^m \int_{w_j \in (\tilde{w}_j^m - \tilde{l}_j^m, \tilde{w}_j^m + \tilde{l}_j^m]} p_j(w_j | d/c_i) dw_j\}. \quad (13)$$

From equalities (8), (9) and (10), one gets

$$\int_{w_j \in (\tilde{w}_j^m - \tilde{l}_j^m, \tilde{w}_j^m + \tilde{l}_j^m]} p_j(w_j | d) dw_j = (\frac{1-d^m}{d^m})(\frac{\tilde{v}_i^m}{\hat{v}_i^m - \tilde{v}_i^m}),$$

and, therefore,

$$\tilde{v}_i^m = d^m\{\hat{v}_i^m(1 - \int_{w_j \in (\tilde{w}_j^m - \tilde{l}_j^m, \tilde{w}_j^m + \tilde{l}_j^m]} p_j(w_j | d) dw_j)$$

$$+ \tilde{v}_i^m \int_{w_j \in (\tilde{w}_j^m - \tilde{l}_j^m, \tilde{w}_j^m + \tilde{l}_j^m]} p_j(w_j | d) dw_j\}. \quad (14)$$

We specify an infinite sequence of strategy profiles $(s^m)_{m=1}^{\infty}$ in the following way. For each $i = 1,2$,

$$s_i^m(h_i^0) = c_i,$$
$$s_i^m(h_i^t) = c_i \text{ if } s_i^m(h_i^{t-1}) = c_i \text{ and } w_i \notin (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m],$$
$$s_i^m(h_i^t) = c_i \text{ if } s_i^m(h_i^{t-1}) = d_i \text{ and } w_i \in (\tilde{w}_i - \tilde{l}_i^m, \tilde{w}_i + \tilde{l}_i^m],$$
$$s_i^m(h_i^t) = d_i \text{ if } s_i^m(h_i^{t-1}) = c_i \text{ and } w_i \in (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m],$$

and

$$s_i^m(h_i^t) = d_i \text{ if } s_i^m(h_i^{t-1}) = d_i \text{ and } w_i \notin (\tilde{w}_i - \tilde{l}_i^m, \tilde{w}_i + \tilde{l}_i^m].$$

Note that there exist $D_i : A \to R$ such that for every $h_i^t \in H_i$ and every $h_j^{t'} \in H_j$,

$$v_i(d^m, s_i^m|_{h_i^t}, s_j^m|_{h_j^{t'}}) = D_i(c) \text{ if } s_i^m(h_i^t) = c_i \text{ and } s_j^m(h_j^{t'}) = c_j,$$

$$v_i(d^m, s_i^m|_{h_i^t}, s_j^m|_{h_j^{t'}}) = D_i(d) \text{ if } s_i^m(h_i^t) = d_i \text{ and } s_j^m(h_j^{t'}) = d_j,$$

$$v_i(d^m, s_i^m|_{h_i^t}, s_j^m|_{h_j^{t'}}) = D_i(c/d_j) \text{ if } s_i^m(h_i^t) = c_i \text{ and } s_j^m(h_j^{t'}) = d_j,$$

and

$$v_i(\boldsymbol{d}^m, s_i^m|_{h_i^t}, s_j^m|_{h_j^{t'}}) = D_i(d/c_j) \text{ if } s_i^m(h_i^t) = d_i \text{ and } s_j^m(h_j^{t'}) = c_j .$$

Note that

$$D_i(c) = 1 - \boldsymbol{d}^m + \boldsymbol{d}^m \{D_i(c) \int_{\substack{w_i \notin (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \notin (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|c)dw + D_i(d) \int_{\substack{w_i \in (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|c)dw$$

$$+ D_i(c/d_j) \int_{\substack{w_i \notin (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|c)dw + D_i(d/c_j) \int_{\substack{w_i \in (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \notin (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|c)dw\},$$

$$D_i(d) = \boldsymbol{d}^m \{D_i(c) \int_{\substack{w_i \in (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \in (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|d)dw + D_i(d) \int_{\substack{w_i \notin (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \notin (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|d)dw$$

$$+ D_i(c/d_j) \int_{\substack{w_i \in (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \notin (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|d)dw + D_i(d/c_j) \int_{\substack{w_i \notin (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \in (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|d)dw\},$$

$$D_i(c/d_j) = (1 - \boldsymbol{d}^m)(-y_i) + \boldsymbol{d}^m \{D_i(c) \int_{\substack{w_i \notin (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \in (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|c/d_j)dw$$

$$+ D_i(d) \int_{\substack{w_i \in (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \notin (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|c/d_j)dw + D_i(c/d_j) \int_{\substack{w_i \notin (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \notin (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|c/d_j)dw$$

$$+ D_i(d/c_j) \int_{\substack{w_i \in (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m] \\ w_j \in (\tilde{w}_j - l_j^m, \tilde{w}_j + l_j^m]}} p(w|c/d_j)dw\},$$

and

$$D_i(d/c_j) = (1 - \boldsymbol{d}^m)(1 + x_i) + \boldsymbol{d}^m \{D_i(c) \int_{\substack{w_i \in (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \notin (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|d/c_j)dw$$

$$+ D_i(d) \int_{\substack{w_i \notin (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|d/c_j)dw + D_i(c/d_j) \int_{\substack{w_i \in (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \in (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|d/c_j)dw$$

$$+ D_i(d/c_j) \int_{\substack{w_i \notin (\tilde{w}_i - l_i^m, \tilde{w}_i + l_i^m] \\ w_j \notin (\hat{w}_j - e_j^m, \hat{w}_j + e_j^m]}} p(w|d/c_j)dw\}.$$

From equalities (11), (12), (13) and (14), one gets that

$$D_i(c) = D_i(d/c_j) = \hat{v}_i^m, \quad D_i(d) = D_i(c/d_j) = \tilde{v}_i^m,$$

and, for every $h_i^t \in H_i$, every $h_j^{t'} \in H_j$, and for every $s_i \in S_i$ satisfying that $s_i|_{(a_i,w_i)} = s_i^m|_{(h_i^t,(a_i,w_i))}$ for all $(a_i, w_i) \in A_i \times \Omega_i$,

$$v_i(\boldsymbol{d}^m, s_i, s_j^m|_{h_j^{t'}}) = \begin{cases} \hat{v}_i^m & \text{if } s_j^m(h_j^{t'}) = c_j \\ \tilde{v}_i^m & \text{if } s_j^m(h_j^{t'}) = d_j \end{cases},$$

and, therefore,

$$v_i(\boldsymbol{d}^m, s_i^m|_{h_i^t}, s_j^m|_{h_j^{t'}}) = v_i(\boldsymbol{d}^m, s_i, s_j^m|_{h_j^{t'}}) . \qquad (15)$$

Equalities (15) imply that for every $h_1^t \in H_1$ and every $\tilde{h}_2^{t'} \in H_2$, $(s_1^m|_{h_1^t}, s_2^m|_{\tilde{h}_2^{t'}})$ is a Nash equilibrium in $\Gamma(\boldsymbol{d}^m)$. Hence, we have completed the proof of Step 1.

Remark: Step 1 offers the following economic implication. Consider the example of the price-setting duopoly presented in Section 2. The state profile "play (c_1, c_2), i.e., play (high price, high price)" is the situation of *price collusion*, while the profile "play (d_1, d_2), i.e., play (low price, low price)" is that of a *price war*. The remaining two profiles, "play (c_1, d_2), i.e., play (high price, low price)", and "play (d_1, c_2), i.e., play (low price, high price)", can be regarded as the situations of a *one-sided secret price cut*. On the equilibrium path sustaining implicit collusion outlined in Step 1, each of these state profiles emerges infinitely many times. This is in contrast with the trigger strategy equilibrium used by Green and Porter (1984) in their study of a quantity-setting duopoly with public monitoring, according to which, both the situation of a price war and the situation of price collusion emerge infinitely many times but the situation of a one-sided secret price cut never emerges.

Step 2: We show that for every positive integer $K > 0$ and every K sustainable payoff vectors $v^{[1]}$, ..., $v^{[K]}$, $\dfrac{\sum_{k=1}^{K} v^{[k]}}{K}$ is also sustainable.

Fix $(\boldsymbol{d}^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty} \boldsymbol{d}^m = 1$. Fix $\boldsymbol{e} > 0$ arbitrarily. For every $k = 1, \ldots, K$, let $(s^{\{k,m\}})_{m=1}^{\infty}$ be an infinite sequence of strategy profiles satisfying that for every large enough $m = 1,2,\ldots,$ $s^{\{k,m\}}$ is a Nash equilibrium in $\Gamma(\boldsymbol{d}^m)$, and

$$v^{[k]} - (\boldsymbol{e}, \boldsymbol{e}) \le \lim_{m\to+\infty} v(\boldsymbol{d}^m, s^{\{k,m\}}) \le v^{[k]} + (\boldsymbol{e}, \boldsymbol{e}) .$$

We define an infinite sequence of strategy profiles $(\overset{=}{s}^m)_{m=1}^{\infty}$ satisfying that

$$\overset{=}{s}_i^m (h_i^{k-1}) = s_i^{\{k,m\}}(h_i^0) ,$$

and for every $t \ge K+1$,

$$\overset{=}{s}_i^m (h_i^{t-1}) = s_i^{\{k,m\}}(\tilde{h}_i^{\tilde{t}}) \text{ if } t = K\tilde{t} + k \text{ and for every } \boldsymbol{t} = 1,\ldots,\tilde{t} ,$$

$$(\tilde{a}_i(\boldsymbol{t}), \tilde{w}_i(\boldsymbol{t})) = (a_i(K\boldsymbol{t} + k), w_i(K\boldsymbol{t} + k)) .$$

Note that

$$\lim_{m\to\infty} v((\boldsymbol{d}^m)^{\frac{1}{K}}, \overset{=}{s}^m) = \lim_{m\to\infty} \frac{\sum_{k=1}^{K} (\boldsymbol{d}^m)^{\frac{k-1}{K}} v(\boldsymbol{d}^m, s^{\{k,m\}})}{\sum_{k=1}^{K} (\boldsymbol{d}^m)^{\frac{k-1}{K}}}$$

$$\in [\frac{\sum_{k=1}^{K} v^{[k]}}{K} - (e,e), \frac{\sum_{k=1}^{K} v^{[k]}}{K} + (e,e)].$$

Since $s^{\{k,m\}}$ is a Nash equilibrium in $\Gamma(d^m)$ for every large enough $m = 1,2,\ldots$, one gets that $\bar{\bar{s}}^m$ is a Nash equilibrium in $\Gamma((d^m)^{\frac{1}{K}})$ for every large enough $m = 1,2,\ldots$. Hence, $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ is sustainable.

Step 3: Note that $(0,0)$ is sustainable, because the repetition of the choices of d is the Nash equilibrium in $\Gamma(d)$ for all $d \in [0,1)$. Step 1 and inequalities (1) imply that $\bar{v}$, $\underline{v}$, $(\bar{v}_1, \underline{v}_2)$ and $(\underline{v}_1, \bar{v}_2)$ are all sustainable. Since the set of sustainable payoff vectors is compact, one gets from Step 2 that the set of sustainable payoff vectors is convex. Hence, every payoff vector in the convex hull of the set $\{(0,0), \bar{v}, (\bar{v}_1, \underline{v}_2), (\underline{v}_1, \bar{v}_2)\}$, i.e., in V^*, is sustainable.

From these observations, we have completed the proof of Theorem 1.

5. The Folk Theorem

This section assumes that players' private signals are *conditionally independent*, i.e.,

$$p(\omega|a) = p_1(\omega_1|a)p_2(\omega_2|a) \text{ for all } a \in A \text{ and all } \omega \in \Omega .$$

A feasible payoff vector $v \in V$ is said to be *individually rational* if it is more than or equal to the minimax payoff vector, i.e., $v \geq (0,0)$. Let

$$z^{[1]} \equiv (0, \frac{1+y_1+x_2}{1+y_1}) \text{ and } z^{[2]} \equiv (\frac{1+y_2+x_1}{1+y_2}, 0) .$$

Note that the set of all feasible and individually rational payoff vectors is equivalent to the convex hull of the set $\{(1,1),(0,0),z^{[1]},z^{[2]}\}$. See Figure 1 again.

We provide the Folk Theorem on the conditional independence assumption in the following way.

Theorem 3: *Suppose that players' private signals are conditionally independent. Then, every feasible and individually rational payoff vector is sustainable.*

We provide the proof of Theorem 3 in the next section.

Theorem 3 is permissive, because we require no informational conditions concerning the accuracy of players' private signals such as the zero likelihood ratio condition. Theorem 3 is in contrast with Matsushima (1990a). Matsushima showed that the repetition of the one-shot Nash equilibrium is the only Nash equilibrium if players' private signals are conditionally independent and only pure strategies are permitted which are restricted to be *independent of payoff-irrelevant histories*. Here, a strategy profile s is said to be independent of payoff-irrelevant histories if for each $i = 1,2$, every $t = 1,2,\ldots$, every $h_i^t \in H_i$, and every $h_i'^t \in H_i$,

$$s_i|_{h_i^t} = s_i|_{h_i'^t} \quad \text{whenever} \quad p_i(h_j^t|s,h_i^t) = p_i(h_j^t|s,h_i'^t) \quad \text{for all } h_j^t \in H_j ,$$

where $p_i(h_j^t|s,h_i^t)$ is the probability anticipated by player i that the opponent j observes private history $h_j^t \in H_j$ when player i observes private history $h_i^t \in H_i$, provided that both players behave according to $s \in S$. Theorem 3 shows that the Folk Theorem holds if players' private signals are conditionally independent and only pure strategies are permitted, but which *depend* on payoff-irrelevant histories.

6. Proof of Theorem 3

The proof of Theorem 3 is divided into four steps.

Step 1: We show that the payoff vectors (1,1), (1,0), (0,1) and (0,0) are all sustainable.

Before constructing Nash equilibria, we consider the situation in which players T times repeatedly play the prisoner-dilemma game. Denote $a^T = (a(1),\dots,a(T))$, $c^T = (c,\dots,c)$, $d^T = (d,\dots,d)$, $(c/d_j)^T = (c/d_j,\dots,c/d_j)$, and so on.

We choose a subset $\Omega_i^* \subset \Omega_i$ satisfying that

$$\int_{\omega_i \in \Omega_i^*} p_i(\omega_i|c)d\omega_i < \int_{\omega_i \in \Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i .$$

We denote by $f_i^*(r,T,a^T)$ the probability that the number of the observed private signals for player i which belong to Ω_i^* is equal to $r \in \{0,\dots,T\}$, conditional that a^T is played. Let $F_i^*(r,T,a^T) \equiv \sum_{r'=0}^{r} f_i^*(r',T,a^T)$. We choose an infinite sequence $(r_i^*(T))_{T=1}^{\infty}$ satisfying that

$$\lim_{T\to\infty} F_i^*(r_i^*(T),T,c^T) = 1, \tag{16}$$

$$\lim_{T\to\infty} \frac{r_i^*(T)}{T} = \int_{\omega_i \in \Omega_i^*} p_i(\omega_i|c)d\omega_i , \tag{17}$$

and

$$\lim_{T\to\infty} Tf_i^*(r_i^*(T),T-1,c^{T-1}) > \frac{1}{\int_{\omega_i \in \Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i - \int_{\omega_i \in \Omega_i^*} p_i(\omega_i|c)d\omega_i} . \tag{18}$$

In the same way as Lemma 1 in Matsushima (1999), one gets that such an infinite sequence $(r_i^*(T))_{T=1}^{\infty}$ exists. The Law of Large Numbers implies that

$$\lim_{T\to\infty} F_i^*(r_i^*(T),T,(c/d_j)^T) = 0 . \tag{19}$$

We choose another subset $\Omega_i^{**} \subset \Omega_i$ satisfying that

$$\int_{\omega_i \in \Omega_i^{**}} p_i(\omega_i|d)d\omega_i < \int_{\omega_i \in \Omega_i^{**}} p_i(\omega_i|d/c_j)d\omega_i .$$

We denote by $f_i^{**}(T,a^T)$ the probability that all of the observed private signals for player i belong to Ω_i^{**}, conditional that a^T is played. Note that

$$\lim_{T\to\infty} \frac{f_i^{**}(T,d^T)}{f_i^{**}(T,(d/c_j)^T)} = 0 . \tag{20}$$

Fix an infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ arbitrarily, which satisfies

$\lim_{m\to\infty}\delta^m = 1$. We choose an infinite sequence of positive integers $(T^m)_{m=1}^{\infty}$ satisfying that

$$\lim_{m\to\infty} T^m = \infty\,,\quad \gamma^m \equiv (\delta^m)^{T^m}\,,\quad \lim_{m\to\infty}\gamma^m = 1\,,$$

and for each $i = 1,2$,

$$\lim_{m\to\Re}\frac{\gamma^m}{1-\gamma^m} f_i^{**}(T^m,(d/c_j)^{T^m}) > y_j\,.$$

Hence, from equalities (16), (19) and (20), we can choose an infinite sequence $(\bar{v}^m,\underline{v}^m,(\bar{\xi}_i^m,\underline{\xi}_i^m)_{i=1,2})_{m=1}^{\infty}$ satisfying that $\bar{\xi}_i^m\in[0,1]$ and $\underline{\xi}_i^m\in[0,1]$ for all $m = 1,2,\ldots$,

$$\lim_{m\to\infty}\bar{v}^m = (1,1)\,,$$

$$\lim_{m\to\infty}\underline{v}^m = (0,0)\,,$$

and for each $i = 1,2$ and every large enough m,

$$\bar{v}_j^m = 1-\frac{\gamma^m}{1-\gamma^m}\bar{\xi}_i^m\{1-F_i^*(r_i^*(T^m),T^m,c^{T^m})\}(\bar{v}_j^m-\underline{v}_j^m)$$

$$= 1+x_j-\frac{\gamma^m}{1-\gamma^m}\bar{\xi}_i^m\{1-F_i^*(r_i^*(T^m),T^m,(c/d_j)^{T^m})\}(\bar{v}_j^m-\underline{v}_j^m)\,, \tag{21}$$

and

$$\underline{v}_j^m = \frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(T^m,d^{T^m})(\bar{v}_j^m-\underline{v}_j^m)$$

$$= -y_j+\frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(T^m,(d/c_j)^{T^m})(\bar{v}_j^m-\underline{v}_j^m)\,. \tag{22}$$

From the continuity of p_i, we can choose two subsets $\Omega_i^*(\xi)\subset\Omega_i$ and $\Omega_i^{**}(\xi)\subset\Omega_i^{**}$ for every $\xi\in[0,1]$ sufficiently close to 1, satisfying that

$$\xi = \frac{\int_{\omega_i\in\Omega_i^*(\xi)\cap\Omega_i^*} p_i(\omega_i|c)d\omega_i}{\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c)d\omega_i} = \frac{\int_{\omega_i\in\Omega_i^*(\xi),\,\omega_i\notin\Omega_i^*} p_i(\omega_i|c)d\omega_i}{\int_{\omega_i\notin\Omega_i^*} p_i(\omega_i|c)d\omega_i}$$

$$= \frac{\int_{\omega_i\in\Omega_i^*(\xi)\cap\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i}{\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i} = \frac{\int_{\omega_i\in\Omega_i^*(\xi),\,\omega_i\notin\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i}{\int_{\omega_i\notin\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i}$$

$$= \frac{\int_{\omega_i\in\Omega_i^{**}(\xi)} p_i(\omega_i|d)d\omega_i}{\int_{\omega_i\in\Omega_i^{**}} p_i(\omega_i|d)d\omega_i} = \frac{\int_{\omega_i\in\Omega_i^{**}(\xi)} p_i(\omega_i|d/c_j)d\omega_i}{\int_{\omega_i\in\Omega_i^{**}} p_i(\omega_i|d/c_j)d\omega_i}\,.$$

These equalities imply that the probability of $\omega_i\in\Omega_i^*(\xi)$ is the same between the case of the choice of action profile c and the case of the choice of action profile c/d_j, the probability of $\omega_i \in \Omega_i^*(\xi)$ conditional on $\omega_i \in \Omega_i^*$ is equivalent to that conditional on $\omega_i \notin \Omega_i^*(\xi)$, and the probability of $\omega_i \in \Omega_i^{**}(\xi)$ conditional on $\omega_i \in \Omega_i^{**}$ is the same between the case of the choice of action profile d and the case of the choice of action profile d/c_j.

For every $m = 1,2,\ldots$, we define two subsets of the T^m times product of Ω_i, $\Phi_i^{*m} \subset \Omega_i^{T^m}$ and $\Phi_i^{**m} \subset \Omega_i^{T^m}$, by

$$\Phi_i^{*m} \equiv \{ (\omega_i(1),\ldots,\omega_i(T^m)) \in \Omega_i^{T^m} : \text{ either } \omega_i(t) \in \Omega_i^* \text{ for at most } r_i^*(T^m) \text{ periods, or } \omega_i(T^m) \notin \Omega_i^*(\bar{\xi}_i^m) \},$$

and

$$\Phi_i^{**m} \equiv \{ (\omega_i(1),\ldots,\omega_i(T^m)) \in \Omega_i^{T^m} : \text{ either } \omega_i(t) \in \Omega_i^{**} \text{ for all } t \in \{1,\ldots,T^m\}, \text{ or } \omega_i(T^m) \notin \Omega_i^{**}(\underline{\xi}_i^m) \}.$$

Based on the above definitions, we consider the following Markov strategies with $2T^m$ states, i.e., with states (c_i,τ) and (d_i,τ) for all $\tau = 1,\ldots,T^m$. When player $i's$ state is state (c_i,τ) (state (d_i,τ)), player i chooses action c_i (action d_i, respectively). When in a period t player $i's$ state is state (c_i,τ) (state (d_i,τ)) and $\tau < T^m$, player $i's$ state in the next period $t+1$ will be state $(c_i,\tau+1)$ (state $(d_i,\tau+1)$, respectively). When in a period t player $i's$ state is (c_i,T^m) and the vector of her private signals observed in the past T^m periods $(\omega_i(t-T^m+1),\ldots,\omega_i(t))$ belongs to Φ_i^{*m} (does not belong to Φ_i^{*m}), player $i's$ state in the next period $t+1$ will be state $(c_i,1)$ (state $(d_i,1)$, respectively). When in a period t player $i's$ state is state (d_i,T^m) and $(\omega_i(t-T^m+1),\ldots,\omega_i(t))$ belongs to Φ_i^{**m} (does not belong to Φ_i^{**m}), player $i's$ state in the next period $t+1$ will be state $(c_i,1)$ (state $(d_i,\tau+1)$, respectively). See Figure 3.

[Figure 3]

We denote by $\bar{s}_i^m$ and $\underline{s}_i^m$ the strategies which start with state $(c_i,1)$ and state $(d_i,1)$, respectively. In order to prove that $\bar{s}^m$, $\underline{s}^m$, $\bar{s}^m/\underline{s}_j^m$ and $\underline{s}^m/\bar{s}_j^m$ are all Nash equilibria, we will make the following two requirements; that for every $k = 0,1,\ldots$, any mixture of the choices of action c_i and action d_i in the T^m times repeated play is less preferable than the T^m times repeated choice of action c_i or the T^m times repeated choice of action d_i in period $t = kT^m + 1$, irrespective of her own private history; and that for every $k = 0,1,\ldots$, each player i is indifferent between $\bar{s}_i^m$ and $\underline{s}_i^m$ in period $t = kT^m + 1$, irrespective of her own private history. Note that the second requirement is similar to that of Step 1 in the proof of Theorem 1. This requirement makes our analysis more complicated than the simple application of the Law of Large Numbers in the study of review strategies by, for example, Radner (1985). However, by using the lemmata in Matsushima (1999), we can prove that both requirements are satisfied when m is large enough.

Formally, we specify an infinite sequence of two strategy profiles $(\bar{s}^m, \underline{s}^m)_{m=1}^{\infty}$ in the following way. For each $i = 1,2$,

$$\bar{s}_i^m(h_i^{t-1}) = c_i \quad \text{and} \quad \underline{s}_i^m(h_i^{t-1}) = d_i \quad \text{for all} \quad t = 1,\ldots,T^m \quad \text{and all} \quad h_i^{t-1} \in H_i,$$

$$\bar{s}_i^m|_{h_i^{T^m}} = \bar{s}_i^m \text{ if } (\omega_i(1),\ldots,\omega_i(T^m)) \in \Phi_i^{*m},$$

$$\bar{s}_i^m|_{h_i^{T^m}} = \underline{s}_i^m \text{ if } (\omega_i(1),\ldots,\omega_i(T^m)) \notin \Phi_i^{*m},$$

$$\underline{s}_i^m|_{h_i^{T^m}} = \bar{s}_i^m \text{ if } (\omega_i(1),\ldots,\omega_i(T^m)) \in \Phi_i^{**m},$$

and

$$\underline{s}_i^m|_{h_i^{T^m}} = \underline{s}_i^m \text{ if } (\omega_i(1),\ldots,\omega_i(T^m)) \notin \Phi_i^{**m}.$$

These strategies are regarded as a modification of the review strategy originated by Radner (1985).[6] When the T^m times repeated play passes the review of player i, that is, either $(\omega_i(1),\ldots,\omega_i(T^m)) \in \Phi_i^{*m}$ in the case of player $i's$ T^m times repeated choice of action c_i or $(\omega_i(1),\ldots,\omega_i(T^m)) \in \Phi_i^{**m}$ in the case of player $i's$ T^m times repeated choice of action d_i, player i will play collusive behavior during the next T^m periods according to $\bar{s}_i^m$. When the T^m times repeated play fails the review of player i, that is, either $(\omega_i(1),\ldots,\omega_i(T^m)) \notin \Phi_i^{*m}$ in the case of player $i's$ T^m times repeated choice of action c_i or $(\omega_i(1),\ldots,\omega_i(T^m)) \notin \Phi_i^{**m}$ in the case of player $i's$ T^m times repeated choice of action d_i, player i will play punishment behavior during the next T^m periods according to $\underline{s}_i^m$.

Equalities (21) and (22) imply that

$$v_j(\delta^m, \bar{s}^m) = v_j(\delta^m, \bar{s}^m / \underline{s}_j^m) = \bar{v}_j^m, \tag{23}$$

---

[6] See also Abreu, Milgrom and Pearce (1991), Matsushima (1999), Kandori and Matsushima (1998), and Compte (1998). These papers made future punishment triggered by either bad histories of the public signals during the review phase or bad messages announced at the last stage of the review phase. In contrast, the present paper assumes the non-existence of such public signals or messages.

$$v_j(\delta^m, \underline{s}^m) = v_j(\delta^m, \underline{s}^m / \bar{s}_j^m) = \underline{v}_j^m, \tag{24}$$

and therefore,

$$\lim_{m\to\infty} v_j(\delta^m, \bar{s}^m) = \lim_{m\to\infty} v_j(\delta^m, \bar{s}^m / \underline{s}_j^m) = 1,$$

and

$$\lim_{m\to\infty} v_j(\delta^m, \underline{s}^m) = \lim_{m\to\infty} v_j(\delta^m, \underline{s}^m / \bar{s}_j^m) = 0.$$

We show below that $\bar{s}^m$, $\underline{s}^m$, $\bar{s}^m / \underline{s}_j^m$ and $\underline{s}^m / \bar{s}_j^m$ are all Nash equilibria for every large enough m. Suppose that there exists $s_j \in S_j$ such that

$$v_j(\delta^m, \bar{s}^m / s_j) > \bar{v}_j^m,$$

and

$$s_j|_{h_j^{T^m}} = \bar{s}_j^m|_{h_j^{T^m}} \text{ for all } h_j^{T^m} \in H_j.$$

The definition of $\Omega_i^*(\bar{\xi}_i^m)$ and the conditional independence assumption imply that we can assume that there exists $(a_j(1),\ldots,a_j(T^m))$ such that

$$s_j(h_j^{t-1}) = a_j(t) \text{ for all } t = 1,\ldots,T^m \text{ and all } h_j^{t-1} \in H_j.$$

In the same way as Lemma 4 in Matsushima (1999), one gets that, given that m is large enough, player j can obtain a positive gain from deviation by choosing action d_j either only in the first period or in all T^m periods. Hence, we can assume that either $a_j(1) = d_j$ and $a_j(t) = c_j$ for all $t = 2,\ldots,T^m$, or $a_j(t) = d_j$ for all $t = 1,\ldots,T^m$. Equalities (23) imply that player j cannot obtain any gain from deviation by choosing action d_j in all T^m periods. Moreover, In the same way as Lemma 5 in Matsushima (1999), we can show that, given that m is large enough, player j cannot obtain any gain from deviation by choosing action d_j only in the first period, as follows. Note that the difference of the probabilities that event Φ_i^{*m} does not occur (i.e., player j is punished) between in the case of $a^{T^m} = (c / d_j, c,\ldots,c)$ and in the case of $a^{T^m} = c^{T^m}$ is equal to

$$\bar{\xi}_i^m\{\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i - \int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c)d\omega_i\}\ f_i^*(r_i^*(T^m), T^m - 1, c^{T^m-1}).$$

Hence, the difference of the (un-normalized) long-run payoffs for player j in the case of the T^m times repeated choice of action c_j and the case of the deviation by choosing action d_j only in the first period is equal to

$$x_j - \bar{\xi}_i^m\{\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i$$

$$-\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c)d\omega_i\}\ f_i^*(r_i^*(T^m), T^m - 1, c^{T^m-1})\,\frac{\gamma^m}{1-\delta^m}\,(\bar{v}_j^m - \underline{v}_j^m).$$

From equality (19), the latter equality of (21), $\lim_{m\to\infty}\gamma^m = \lim_{m\to\infty}(\delta^m)^{T^m} = 1$, $\lim_{m\to\infty}\bar{v}^m = (1,1)$

and $\lim_{m\to\infty} \underline{v}^m = (0,0)$, one gets that

$$\lim_{m\to\infty} \frac{\gamma^m}{(1-\delta^m)T^m}\bar{\xi}_i^m = \lim_{m\to\infty} \frac{\gamma^m}{1-\gamma^m}\bar{\xi}_i^m \lim_{m\to\infty} \frac{\sum_{t=0}^{T^m-1}(\delta^m)^t}{T^m} = x_j,$$

and therefore, the limit of this difference in long-run payoffs is equal to

$$x_j - x_j\{\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c/d_j)d\omega_i$$
$$-\int_{\omega_i\in\Omega_i^*} p_i(\omega_i|c)d\omega_i\} \lim_{T\to\infty} Tf_i^*(r_i^*(T), T-1, c^{T-1}),$$

which is less than zero, because of inequality (18). Hence, player j have no incentive to deviate by choosing action d_j only in the first period. This, however, is a contradiction.

Next, suppose that there exists $s_j \in S_j$ such that

$$v_j(\delta^m, \underline{s}^m / s_j) > \underline{v}_j^m,$$

and

$$s_j|_{h_j^{T^m}} = \underline{s}_j^m|_{h_j^{T^m}} \text{ for all } h_j^{T^m} \in H_j.$$

The definition of $\Omega_i^*(\bar{\xi}_i^m)$ and the conditional independence assumption imply that we can assume that there exists $(a_j(1),\ldots,a_j(T^m))$ such that

$$s_j(h_j^{t-1}) = a_j(t) \text{ for all } t = 1,\ldots,T^m \text{ and all } h_j^{t-1} \in H_j.$$

Let $\tau \in \{1,\ldots,T^m\}$ denote the number of the periods in which $a_j(t) = c_j$. Without loss of generality, we can assume that player j chooses action c_j in the last τ periods, i.e., from period $T^m - \tau + 1$ to period T^m. Note that

$$f_i^{**}(T^m, a^{T^m}) = (\int_{\omega_i\in\Omega_i^{**}} p_i(\omega_i|d/c_j)d\omega_i)^\tau (\int_{\omega_i\in\Omega_i^{**}} p_i(\omega_i|d)d\omega_i)^{T^m-\tau}$$
$$= q^\tau f_i^{**}(T^m, d^{T^m}),$$

where

$$q \equiv \frac{\int_{\omega_i\in\Omega_i^{**}} p_i(\omega_i|d/c_j)d\omega_i}{\int_{\omega_i\in\Omega_i^{**}} p_i(\omega_i|d)d\omega_i} > 1.$$

Hence, one gets from equalities (22) that

$$v_j(\delta^m, \underline{s}^m / s_j)$$
$$= \frac{1-\delta^m}{1-\gamma^m}(-y_j)\sum_{t=T^m-\tau}^{T^m-1}(\delta^m)^t + \frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(T^m, a^{T^m})(\bar{v}_j^m - \underline{v}_j^m)$$
$$= \frac{1-\delta^m}{1-\gamma^m}(-y_j)\sum_{t=T^m-\tau}^{T^m-1}(\delta^m)^t + q^\tau \underline{v}_j^m.$$

We denote by $v_j(\tau)$ the right hand side of these equalities, i.e.,

$$v_j(\tau) \equiv \frac{1-\delta^m}{1-\gamma^m}(-y_j)\sum_{t=T^m-\tau}^{T^m-1}(\delta^m)^t + q^\tau \underline{v}_j^m .$$

Note that

$$v_j(\tau+1)-v_j(\tau) = (\frac{1}{\delta^m})^\tau\{\frac{(\delta^m)^{T^m-1}-\gamma^m}{1-\gamma^m}(-y_j)\} + q^\tau(q-1)\underline{v}_j^m . \tag{25}$$

Note also that

$$v_j(0) = v_j(\delta^m, \underline{s}^m) = v_j(\delta^m, \bar{s}^m) = v_j(T^m) . \tag{26}$$

Given that m is large enough, we can assume that

$$1 < \frac{1}{\delta^m} < q .$$

This, together with equality (25), inequality $\frac{(\delta^m)^{T^m-1}-\gamma^m}{1-\gamma^m}(-y_j) < 0$, and inequality $(q-1)\underline{v}_j^m > 0$, implies that for every $\tau \in \{1,...,T^m\}$,

$$v_j(\tau+1)-v_j(\tau) \geq 0 \text{ if } v_j(\tau)-v_j(\tau-1) \geq 0 ,$$

and

$$v_j(\tau)-v_j(\tau-1) \leq 0 \text{ if } v_j(\tau+1)-v_j(\tau) \leq 0 .$$

This, together with equalities (26), implies that

$$v_j(0) \geq v_j(\tau) \text{ for all } \tau \in \{1,...,T^m\} .$$

Hence, it must hold that $v_j(\delta^m, \underline{s}^m / s_j) \leq \underline{v}_j^m$, but this is a contradiction.

From these observations, we have proved that $\bar{s}^m$, $\underline{s}^m$, $\bar{s}^m / \underline{s}_j^m$ and $\underline{s}^m / \bar{s}_j^m$ are all Nash equilibria for every large enough m. Hence, (1,1), (1,0), (0,1) and (0,0) are all sustainable.

Step 2: We show that $z^{[1]}$ and $z^{[2]}$ are both sustainable. Consider $z^{[1]}$ only. We can prove that $z^{[2]}$ is sustainable in the same way.

Before constructing Nash equilibria, we consider the situation in which players M times repeatedly play the prisoner-dilemma game. We choose a subset $\Omega_2^+ \subset \Omega_2$ satisfying that

$$\int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d/c_1)d\omega_2 < \int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d)d\omega_2 .$$

We denote by $f_2^+(r, M, a^M)$ the probability that the number of the observed private signals for player 2 which belong to Ω_2^+ is equal to $r \in \{0,\dots,M\}$, conditional that a^M is played. Let $F_2^+(r, M, a^M) \equiv \sum_{r'=0}^{r} f_2^+(r', M, a^M)$. We choose an infinite sequence

$(r_2^+(M))_{M=1}^{\infty}$ satisfying that

$$\lim_{M\to\infty} F_2^+(r_2^+(M), M, (d/c_1)^M) = 1, \tag{27}$$

$$\lim_{M\to\infty} \frac{r_2^+(M)}{M} = \int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d/c_1)d\omega_2 , \tag{28}$$

and

$$\lim_{M\to\infty} Mf_2^+(r_2^+(M), M, (d/c_1)^M)$$
$$> \frac{1+y_1}{y_1\{\int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d)d\omega_2 - \int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d/c_1)d\omega_2\}} \tag{29}$$

In the same way as Lemma 1 in Matsushima (1999), one gets that such an infinite sequence $(r_2^+(M))_{M=1}^{\infty}$ exists. The Law of Large Numbers implies that

$$\lim_{M\to\infty} F_2^+(r_2^+(M), M, d^M) = 0 . \tag{30}$$

We choose a positive real number $b > 0$ arbitrarily, which is less than but close to

$$\frac{1}{1+y_1},$$

satisfying that

$$\lim_{M\to\infty} Mf_2^+(r_2^+(M), M, (d/c_1)^M)$$
$$> \frac{by_1}{(1-b)^2\{\int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d)d\omega_2 - \int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d/c_1)d\omega_2\}} .$$

Let

$$v^* \equiv b(-y_1, 1+x_2) + (1-b)(1,1) .$$

Note that v^* approximates $z^{[1]}$, and

$$v_1^* > z_1^{[1]} = 0 .$$

Fix an infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty} \delta^m = 1$. We choose an infinite sequence of positive integers $(M^m)_{m=1}^{\infty}$ satisfying that

$$\lim_{m\to\infty} M^m = \infty , \quad \chi^m \equiv (\delta^m)^{M^m} ,$$

and

$$\lim_{m\to\infty} \chi^m = 1-b . \tag{31}$$

For every $m = 1,2,\ldots$, we define a subset of the M^m times product of Ω_2, $\Phi_2^{+m} \subset \Omega_2^{M^m}$, by

$$\Phi_2^{+m} \equiv \{ (\omega_2(1),\ldots,\omega_2(M^m)) \in \Omega_2^{M^m} : \ \omega_2(t) \in \Omega_2^+ \text{ for at most } r_2^+(M^m) \text{ periods}\}.$$

Let $(\bar{s}^m, \underline{s}^m)_{m=1}^{\infty}$ be the infinite sequence of the two strategy profiles specified in Step 1.

Consider the following strategy profile. In the first M^m periods, player 1 always chooses action c_1 and player 2 always chooses action d_2. From period M^m+1, player 1 certainly plays the strategy $\bar{s}_1^m$. From period M^m+1, player 2 plays strategy $\bar{s}_2^m$ (strategy $\underline{s}_2^m$) if the vector of the observed private signals $(\omega_2(1),\ldots,\omega_2(M^m))$ passes the review, i.e., belongs to Φ_2^+ (fails the review, i.e., does not belong to Φ_2^+, respectively). See Figures 4.1 and 4.2.

[Figure 4.1]

[Figure 4.2]

Formally, we specify an infinite sequence of strategy profiles $(s^{[1,m]})_{m=1}^{\infty}$ in the following way.

$$s^{[1,m]}(h_2^{t-1})=(c_1,d_2) \text{ if } 1\le t\le M^m,$$

$$s_1^{[1,m]}|_{h_1^{T^m}}=\bar{s}_1^m \text{ for all } h_1^{T^m},$$

$$s_2^{[1,m]}|_{h_2^{T^m}}=\bar{s}_2^m \text{ if } (\omega_2(1),\ldots,\omega_2(M^m))\in\Phi_2^+,$$

and

$$\bar{s}_2^{[1,m]}|_{h_2^{T^m}}=\underline{s}_2^m \text{ if } (\omega_2(1),\ldots,\omega_2(M^m))\notin\Phi_2^+.$$

Note that

$$\begin{aligned}&v_1(\delta^m,s^{[1,m]})\\&=(1-\chi^m)(-y_1)+\chi^m\{F_2^+(r_2^+(M^m),M^m,(d/c_1)^{M^m})\bar{v}_1^m\\&+(1-F_2^+(r_2^+(M^m),M^m,(d/c_1)^{M^m}))\underline{v}_1^m\},\end{aligned}$$

and

$$v_2(\delta^m,s^{[1,m]})=(1-\chi^m)(1+x_2)+\chi^m\bar{v}_2^m.$$

Note from equalities (27), (30) and (31) that

$$\lim_{m\to\infty}v(\delta^m,s^{[1,m]})=b(-y_1,1+x_2)+(1-b)(1,1)=v^*.$$

Hence, $v(\delta^m,s^{[1,m]})$ approximates $z^{[1]}$ for every large enough m.

We show below that $s^{[1,m]}$ is a Nash equilibrium for every large enough m. Step 1 and the definition of $s^{[1,m]}$ imply that $(s_1^{[1,m]}|_{h_1^{M^m}},s_2^{[1,m]}|_{h_2^{M^m}})$ is a Nash equilibrium for every $(h_1^{M^m},h_2^{M^m})$ and every large enough m. Since players' private signals are conditionally independent and action d_2 is dominant in the component game, the repeated choice of action d_2 during the first M^m periods is the best response for player 2. Hence, all we have to do is to check that the repeated choice of action c_1 during the first M^m periods is the best response for player 1 for every large enough m.

Suppose that there exists $s_1 \in S_1$ such that

$$u_1(s_1, s_2^{[1,m]}) > u_1(s^{[1,m]}),$$

and

$$s_1|_{h_1^{M^m}} = s_1^{[1,m]}|_{h_1^{M^m}} \text{ for all } h_1^{M^m} \in H_1.$$

From the conditional independence assumption, we can assume that there exists $(a_1(1),\dots,a_1(M^m))$ such that

$$s_1(h_1^{t-1}) = a_1(t) \text{ for all } t = 1,\dots,M^m \text{ and all } h_1^{t-1} \in H_1.$$

In the same way as Lemma 4 in Matsushima (1999), one gets that, given that m is large enough, player 1 can obtain a positive gain from deviation by choosing action d_1 either only in the first period or in all these M^m periods. Hence, we can assume that either $a_1(1) = d_1$ and $a_1(t) = c_1$ for all $t = 2,\dots,M^m$, or $a_1(t) = d_1$ for all $t = 1,\dots,M^m$. In the same way as Lemma 5 in Matsushima (1999) and Step 1 in the proof of this theorem, we can show that, given that m is large enough, player 1 cannot obtain any gain from deviation by choosing action d_1 only in the first period, as follows. Note that the difference of the probabilities that event Φ_i^{+m} does not occur (i.e., player j is punished) between in the case of $a^{T^m} = (d, d/c_1,\dots,d/c_1)$ and in the case of $a^{T^m} = (d/c_1)^{T^m}$ is equal to

$$\{\int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d)d\omega_2 - \int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d/c_1)d\omega_2\}\ f_2^+(r_2^+(M^m), M^m-1, (d/c_1)^{M^m-1}).$$

Hence, the difference of the long-run payoffs for player 1 in the case of the T^m times repeated play of action c_1 and the case of the deviation by choosing action d_1 only in the first period is equal to

$$y_1 - \{\int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d)d\omega_2 - \int_{\omega_2\in\Omega_2^+} p_2(\omega_2|d/c_1)d\omega_2\}$$

$$\cdot\ f_2^+(r_2^+(M^m), M^m-1, (d/c_1)^{M^m-1})\ \frac{\chi^m}{1-\delta^m}(\bar{v}_1^m - \underline{v}_1^m).$$

From equality (31), one gets that

$$\lim_{m\to\infty}\frac{\chi^m}{(1-\delta^m)M^m} = \lim_{m\to\infty}\frac{\chi^m}{1-\chi^m}\lim_{m\to\infty}\frac{\sum_{t=0}^{M^m-1}(\delta^m)^t}{M^m} = \frac{(1-b)^2}{b},$$

which, together with $\lim_{m\to\infty}\bar{v}^m = (1,1)$ and $\lim_{m\to\infty}\underline{v}^m = (0,0)$, implies that the limit of this difference in long-run payoffs is equal to

$$y_1 - \{ \int_{\omega_2 \in \Omega_2^+} p_2(\omega_2 | d) d\omega_2 - \int_{\omega_2 \in \Omega_2^+} p_2(\omega_2 | d / c_1) d\omega_2 \}$$

$$\cdot \frac{(1-b)^2}{b} \lim_{M \to \infty} M f_2^+(r_2^+(M), M-1, (d/c_1)^{M-1}) .$$

Since b is close to $\frac{1}{1+y_1}$, this value is approximated by

$$y_1 - \{ \int_{\omega_2 \in \Omega_2^+} p_2(\omega_2 | d) d\omega_2 - \int_{\omega_2 \in \Omega_2^+} p_2(\omega_2 | d / c_1) d\omega_2 \}$$

$$\cdot \frac{{y_1}^2}{1+y_1} \lim_{M \to \infty} M f_2^+(r_2^+(M), M-1, (d/c_1)^{M-1}) ,$$

which is less than zero, because of inequality (29). Hence, player 1 have no incentive to deviate by choosing action d_1 only in the first period, when m is large enough. This, however, is a contradiction. Hence, it must hold that $a_1(t) = d_1$ for all $t = 1, \ldots, M^m$. Note from $\lim_{m \to \infty} \underline{v}^m = (0,0)$ and equality (30) that if $a_1(t) = d_1$ for all $t = 1, \ldots, M^m$, then

$$\lim_{m \to \infty} v_1(\delta^m, (s_1, s_1^{[1,m]}))$$

$$= \lim_{m \to \infty} \chi^m \{ F_2^+(r_2^+(M^m), M^m, d^{M^m}) \bar{v}_1^m$$

$$+ (1 - F_2^+(r_2^+(M^m), M^m, d^{M^m})) \underline{v}_1^m \} = 0 .$$

Since $\lim_{m \to \infty} v_1(\delta^m, s^{[1,m]}) = v_1^* > 0$, one gets that $v_1(\delta^m, (s_1, s_2^{[1,m]})) < v_1(\delta^m, s^{[1,m]})$ for every large enough m, but this is a contradiction.

Hence, we have proved that $z^{[1]}$ is sustainable. Similarly, $z^{[2]}$ is sustainable too.

Step 3: Step 2 in the proof of Theorem 1 has proved that for every positive integer $K > 0$ and every K sustainable payoff vectors $v^{[1]}$, ..., $v^{[K]}$, $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ is also sustainable.

Step 4: Step 1 and Step 2 imply that (1,1), (0,0), $z^{[1]}$ and $z^{[2]}$ are all sustainable. Since the set of sustainable payoff vectors is compact, one gets from Step 3 that the set of sustainable payoff vectors is convex. Hence, every payoff vector in the convex hull of the set $\{(0,0), (1,1), z^{[1]}, z^{[2]}\}$, i.e., every feasible and individually rational payoff vector, is sustainable.

From these observations, we have completed the proof of Theorem 3.

7. Uniform Sustainability

In contrast with Sections 5 and 6, this section require no presumptions on the private signal structure such as conditional indifference. This section regards $\Gamma(d)$ as a machine game.[7] For each $i = 1,2$, fix the finite set of states of machine for player i, Q_i, arbitrarily, where $|Q_i| \geq 2$. Let $Q \equiv Q_1 \times Q_2$. A *rule* for player i is defined by $s_i \equiv (f_i, t_i)$, where $f_i : Q_i \to A_i$ is an output function, $t_i : Q_i \times \Omega_i \to Q_i$ is a transition function, and t_i is measurable w. r. t. Ω_i. The set of rules for player i is denoted by Σ_i. Let $\Sigma \equiv \Sigma_1 \times \Sigma_2$. A *machine for player* i is defined as a combination of a rule and an initial state, $q_i = (s_i, q_i) \in \Sigma_i \times Q_i$. In every period t, player i chooses action $a_i(t) = f_i(q_i(t))$ where $q_i(t)$ is the state for player i in period t. The state for player i will transit from $q_i(t)$ to $q_i(t+1) = t_i(q_i(t), w_I(T))$ in period $t+1$ when she observes private signal $w_i(t)$ in period t. Player $i's$ normalized long-run payoff induced by a machine profile $q \in \Theta$ is defined by $v_i(d, q) \equiv (1-d) E[\sum_{t=1}^{\infty} d^{t-1} u_i(a(t)) | q]$. The set of all machines for player i is denoted by Θ_i. Let $\Theta \equiv \Theta_1 \times \Theta_2$. A machine profile $q \in \Theta$ is said to be a *Nash equilibrium* in $\Gamma(d)$ if for each $i = 1,2$ and every $q_i' \in \Theta_i$, $v_i(d_i, q) \geq v_i(d_i, q / q_i')$. A machine profile $q \in \Theta$ is sometimes denoted by $(s, q) \in \Sigma \times Q$.

For every $i = 1,2$ and every machine $q_i = (s_i, q_i) \in \Theta_i$ for player i, we define a strategy $s_i(q_i) \in S_i$ for player i and a function $q_i(q_i) : H_i \to Q_i$ by

$$q_i(q_i)(h_i^0) = q_i \in Q_i,$$

$$s_i(q_i)(h_i^0) = f_i(q_i) \in A_i,$$

and for every $t \geq 1$ and every $h_i^t \in H_i$,

$$q_i(q_i)(h_i^t) = t_i(q_i(q_i)(h_i^{t-1}), w_i(t)) \in Q_i,$$

and

$$s_i(q_i)(h_i^t) = f_i(q_i(q_i)(h_i^t)) \in A_i.$$

Let $s(q) \equiv (s_1(q_1), s_2(q_2)) \in S$. Note that q is a Nash equilibrium in $\Gamma(d)$ if $s(q)$ is a Nash equilibrium in $\Gamma(d)$. Note also that $v(d, q) = v(d, s(q))$.

We assume that players' initial states are not common knowledge. We introduce the following solution concept of a rule profile.

Definition 2: A rule profile $s \in \Sigma$ is a *uniform equilibrium* in $\Gamma(d)$ if (s, q) is a Nash equilibrium in $\Gamma(d)$ for all $q \in Q$.

---

[7] For the definition of a machine game in the perfect monitoring case, see Rubinstein (1994, Chapter 9). This section extends this definition to the private monitoring case.

Definition 2 requires that players always play Nash equilibria irrespective of their initial states. This means that all possible Nash equilibria are interchangeable.

Note that s is a uniform equilibrium in $\Gamma(d)$ if $s(s,q)$ is a Nash equilibrium in $\Gamma(d)$ for all $q \in Q$. Note also that

$$v(d,(s,q)) = v(d,s(s,q)) \text{ for all } q \in Q.$$

Note that if s is a uniform equilibrium in $\Gamma(d)$, then, for each $i = 1,2$, every $q \in Q$ and every $q_i' \in Q_i$,

$$v_i(d,(s,q)) = v_i(d,(s,q/q_i')),$$

that is, each player can obtain the same payoff irrespective of her own initial state. However, $v_i(d,(s,q))$ is not necessarily equivalent to $v_i(d,(s,q/q_j'))$ for every $q \in Q$ and every $q_j' \in Q_j$, i.e., the payoff which each player obtains may depend on her opponent's initial state.

The following theorem states that for every discount factor there exist a uniform equilibrium and a state profile, the combination of which sustains the payoff vector Pareto-dominating all other payoff vectors induced by the machine profiles consistent with the other uniform equilibria.

Theorem 4: *For every $d \in [0,1)$, there exists a uniform equilibrium $s \in \Sigma$ in $\Gamma(d)$ and $q \in Q$ such that for every uniform equilibrium $s' \in \Sigma$ in $\Gamma(d)$ and every $q' \in Q$,*

$$v(d,(s,q)) \geq v(d,s(s',q')).$$

Proof: Fix $d \in [0,1)$ arbitrarily, and consider a uniform equilibrium $s \in \Sigma$ in $\Gamma(d)$. Note that for every $q \in Q$,

$$v(d,s,q) \geq 0,$$

because each player i obtains at least payoff zero by always choosing action d_i.

Suppose that there exists $i = 1,2$ such that $f_i(q_i) = d_i$ for all $q_i \in Q_i$. Since the choice of action d_j is the dominant action for player j in the component game G, it must hold that

$$f_j(q_j) = d_j \text{ for all } q_j \in Q_j.$$

Hence, one gets that both players repeatedly choose this dominant action profile d, and therefore,

$$v(d,s,q) = 0.$$

Next, suppose that for each $i = 1,2$, there exists $q_i \in Q_i$ such that $f_i(q_i) = c_i$. We can check that for each $i = 1,2$ there also exists $q_i' \in Q_i$ such that $f_i(q_i') = d_i$. Otherwise, the only best response for her opponent j is to always choose action d_j irrespective of her state, but this is a contradiction. Hence, one gets from the definition of uniform equilibrium that each player i is always indifferent between the choices of c_i and d_i, and therefore, she is indifferent among all machines, i.e.,

$$v(d,s,q) = v(d,(s,q)/q_i) \text{ for all } q_i \in \Theta_i \text{ and all } q \in Q.$$

This implies that for every $s \in \Sigma$, every $s' \in \Sigma$, every $q \in Q$ and every $q' \in Q$, if s and s' are both uniform equilibria in $\Gamma(d)$, then (s_1, s_2') is also a uniform equilibrium and satisfies

$$v(d,(s_1,s_2'),(q_1,q_2')) = (v_1(d,s,q), v_2(d,s',q')).$$

These observations and the compactness of the set of uniform equilibria imply that there exist a uniform equilibrium $s \in \Sigma$ and $q \in Q$ such that for every uniform equilibrium $s' \in \Sigma$ and every $q' \in Q$, $v(d,(s,q)) \geq v(d, s(s',q'))$.

Q.E.D.

Theorem 4 is in sharp contrast with the fact that there exist a continuous/countable set of Pareto-undominated perfect equilibrium payoff vectors. The following theorem provides an upper-bound of all payoff vectors sustained by machine profiles consistent with uniform equilibria.

Theorem 5: *If* $s \in \Sigma$ *is a uniform equilibrium in* $\Gamma(d)$, *then for each* $i = 1,2$,

$$\max[0, \bar{v}_i] \geq v_i(d,s,q) \text{ for all } q \in Q.$$

Proof: Suppose that there exists $i = 1,2$ such that $f_i(q_i) = d_i$ for all $q_i \in Q_i$. Since d is the dominant action profile, it must hold that

$$f_j(q_j) = d_j \text{ for all } q_j \in Q_j.$$

Hence, players repeatedly choose d, and therefore,

$$v(d,s,q) = 0.$$

Suppose that for each $i = 1,2$, there exists $\tilde{q}_i \in Q_i$ such that $f_i(\tilde{q}_i) = c_i$. Fix $i = 1,2$ arbitrarily, and let

$$W_i(q_j) \equiv \max_{q_i' \in \Theta_i} v_i(d_i, q_i', (s_j, q_j)).$$

The uniform equilibrium property of s implies that for every $q \in Q$,

$$v_i(d,s,q) = W_i(q_j),$$

and therefore,

$$W_i(q_j) = (1-d)u_i(c_i, f_j(q_j))$$
$$+ d \int_{w_j \in \Omega_j} p_j(w_j | c_i, f_j(q_j)) W_i(t_j(q_j, w_j)) dw_j.$$

Choose $q_j^* \in Q_j$ which maximizes $W_i(q_j)$, and suppose that $W_i(q_j^*) > 0$. Note that

$$W_i(q_j^*) = u_i(c_i, f_j(q_j^*))$$

$$+\frac{d}{1-d}\int_{w_j\in\Omega_j} p_j(w_j|c_i, f_j(q_j^*))\{W_i(t_j(q_j^*,w_j)) - W_i(q_i^*)\}dw_j \ .$$

Since $W_i(t_j(q_j^*,w_j)) - W_i(q_i^*) \leq 0$ for all $w_j \in \Omega_j$, one gets that $W_i(q_i^*)$ is less than or equal to the value induced by the following conditional maximization.

$$\max_{\substack{e:\Omega_j\to R_+\cup\{0\},\\ a_j\in A_j}} \{u_i(c_i,a_j) - \int_{w_j\in\Omega_j} p_j(w_j|c_i,a_j)e(w_j)dw_j\}$$

subject to

$$u_i(c_i,a_j) - \int_{w_j\in\Omega_j} p_j(w_j|c_i,a_j)e(w_j)dw_j$$

$$\geq u_i(d_i,a_j) - \int_{w_j\in\Omega_j} p_j(w_j|d_i,a_j)e(w_j)dw_j \ .$$

Since $W_i(q_j^*) > 0$ and $u_i(c/d_j) = -y_i < 0$, one gets that $a_j = c_j$ must hold. Hence, the value induced by the above conditional maximization is equivalent to

$$\max_{e:\Omega_j\to R_+\cup\{0\}} \{1 - \int_{w_j\in\Omega_j} p_j(w_j|c)e(w_j)dw_j\}$$

subject to

$$\int_{w_j\in\Omega_j} \{p_j(w_j|d_i,c_j) - p_j(w_j|c)\}e(w_j)dw_j \geq x_i \ .$$

The value induced by this conditional maximization is equal to

$$1-\frac{\underline{L}_i(c,c/d_j)x_j}{1-\underline{L}_i(c,c/d_j)},$$

which is equal to $\bar{v}_i$.

Q.E.D.

We show below that the upperbound provided by Theorem 5 is the least upperbound. We also show below that there exists a uniform equilibrium such that this upperbound is approximately sustained by every machine profile consistent with it. We introduce the notion of uniform sustainability as follows.

Definition 3: A payoff vector $(v_1,v_2) \in R^2$ is *uniformly sustainable* if for every $(d^m)_{m=1}^{\infty}$ satisfying $\lim_{m\to+\infty} d^m = 1$, and for every $e > 0$, there exists an infinite sequence of rule profiles $(s^m)_{m=1}^{\infty}$ such that for every large enough m, s^m is a uniform equilibrium in $\Gamma(d)$, and for every $q \in Q$,

$$v - (e,e) < \lim_{m\to+\infty} v(d^m, s^m, q) < v + (e,e).$$

Uniform sustainability requires that players always play Nash equilibria irrespective of their initial states and always obtain virtually the same payoff vector irrespective of their initial states, that is, all possible machine profiles are interchangeable and virtually payoff-equivalent Nash equilibria. Note that the set of uniformly sustainable payoff vectors is compact.

Theorem 6: *If* $\bar{v} > \underline{v}$ *and* $v = (v_1, v_2) \in R^2$ *satisfies*

$$\bar{v} \geq v \geq \underline{v},$$

then it is uniformly sustainable.

Proof: Fix $v^+ \in V^*$ and $v^- \in V^*$ arbitrarily, which satisfies $\bar{v} \geq v^+ > v^- \geq \underline{v}$. Fix an infinite sequence of discount factors $(d^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m \to +\infty} d^m = 1$. Let $(s^m)_{m=1}^{\infty}$ be the infinite sequence of strategy profiles specified in Step 1 in the proof of Theorem 1. Let $Q_i = \{q_{i,1}, q_{i,2}, ..., q_{i,b_i}\}$, where $b_i \equiv |Q_i|$. We define an infinite sequence of rule profiles $(s^m)_{m=1}^{\infty}$ in the following way. For each $i = 1,2$,

$$f_i^m(q_{i,1}) = c_i,$$

$$f_i^m(q_i) = d_i \text{ for all } q_i \neq q_{i,1},$$

$$t_i^m(q_{i,1}, w_i) = q_{i,1} \text{ if } w_i \notin (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m],$$

$$t_i^m(q_{i,1}, w_i) = q_{i,2} \text{ if } w_i \in (\hat{w}_i - e_i^m, \hat{w}_i + e_i^m],$$

and for every $q_i \neq q_{i,1}$,

$$t_i^m(q_i, w_i) = q_{i,1} \text{ if } w_i \in (\tilde{w}_i - \tilde{l}_i^m, \tilde{w}_i + \tilde{l}_i^m],$$

and

$$t_i^m(q_i, w_i) = q_{i,2} \text{ if } w_i \notin (\tilde{w}_i - \tilde{l}_i^m, \tilde{w}_i + \tilde{l}_i^m],$$

where $\hat{w}_i$, $\tilde{w}_i$, e_i^m, and l_i^m were specified in Step 1 in the proof of Theorem 1. Note that

$$s_i(s_i^m, q_{i,1}) = s_i^m|_{h_i^t} \text{ if } s_i^m(h_i^t) = c_i,$$

and for every $q_i \neq q_{i,1}$,

$$s_i(s_i^m, q_i) = s_i^m|_{h_i^t} \text{ if } s_i^m(h_i^t) = d_i.$$

Since $(s_1^m|_{h_1^t}, s_2^m|_{h_2^{t'}})$ is a Nash equilibrium in $\Gamma(d^m)$ for every $h_1^t \in H_1$ and every $h_2^{t'} \in H_2$, one gets that $s(s^m, q)$ is a Nash equilibrium in $\Gamma(d^m)$ for all $q \in Q$, and therefore, s^m is a uniform equilibrium in $\Gamma(d^m)$. Since for each $i = 1,2$,

$$v_i(d, (s^m, q)) = v_i(d, s(s^m, q)) = \hat{v}_i^m \text{ if } q_i = q_{i,1},$$

$$v_i(d, (s^m, q)) = v_i(d, s(s^m, q)) = \tilde{v}_i^m \text{ if } q_i \neq q_{i,1},$$

and we can choose v^- as close to v^+ as possible, we have proved that v^+ is uniformly sustainable. Since the set of uniformly sustainable payoff vectors is compact, we have proved that every v satisfying $\bar{v} \geq v \geq \underline{v}$ is uniformly sustainable.

Q.E.D.

Theorems 5 and 6 imply that $\bar{v}$ *is the unique uniformly sustainable payoff vector which Pareto-dominates all other uniformly sustainable payoff vectors*.

Theorem 7: *Suppose that for each* $i = 1,2$, *inequality (3) holds, i.e.,*

$$\underline{L}_i(d, d / c_j) < \frac{1}{1 + y_i}.$$

Then, $(1,1)$ *is uniformly sustainable if and only if for each* $i = 1,2$, *equality (2) holds, i.e.,*

$$\underline{L}_i(c, c / d_j) = 0.$$

Proof: We show the "if" part. Theorem 6, the definition of $\bar{v}$, and equalities (2) imply that if $(1,1) \geq v > \underline{v}$, then v is uniformly sustainable. Inequalities (3) and the definition of $\underline{v}$ imply

$$(1,1) > \underline{v}.$$

Hence, $(1,1)$ is uniformly sustainable.

We show the "only if" part. Theorem 5 implies that for each $i = 1,2$,

$$\max[0, \bar{v}_i] \geq 1.$$

Hence, $\bar{v} = (1,1)$ must hold, which implies equalities (2).

Q.E.D.

Theorem 7 implies that *the efficient payoff vector* $(1,1)$ *is uniformly sustainable if and only if the zero likelihood ratio condition holds, and implies also that if this efficient payoff vector is uniformly sustainable, then it Pareto-dominates all other uniformly sustainable payoff vectors.*

8. Limited Knowledge on the Signal Structure

This section investigates the situation in which players have limited knowledge on their private signal structures. Each player $i=1,2$ knows her own monitoring ability, i.e., knows p_i, but does not know her opponent's monitoring ability, i.e., does not know p_j, and therefore, behaves according to a strategy which does not depend on p_j. All notations in the previous sections will be rewritten as being parameterized by p, if necessary. For example, we write $\overline{v}^{[p]}=(\overline{v}_1^{[p_2]},\overline{v}_2^{[p_1]})$ and $\underline{v}^{[p]}=(\underline{v}_1^{[p_2]},\underline{v}_2^{[p_1]})$ instead of $\overline{v}=(\overline{v}_1,\overline{v}_2)$ and $\underline{v}=(\underline{v}_1,\underline{v}_2)$, respectively.

8.1. The Folk Theorem

We reconsider sustainability by Nash equilibrium. For each $i=1,2$, fix an arbitrary compact and nonempty subset P_i^* of conditional density functions on player $i's$ private signal. Let $P^*\equiv P_1^*\times P_2^*$. We assume that each player i only knows which element of P_i^* is the correct conditional density function for her own private signal. We assume that it is common knowledge that the correct conditional density function belongs to P^*. We assume also that it is common knowledge that players' private signals are conditionally independent. A mapping assigning each element of P_i^* a strategy for player i is denoted by $\boldsymbol{r}_i\!:P_i^*\to S_i$. Let $\boldsymbol{r}\equiv(\boldsymbol{r}_1,\boldsymbol{r}_2)$, and $\boldsymbol{r}(p)\equiv(\boldsymbol{r}_1(p_1),\boldsymbol{r}_2(p_2))$. Player i plays the assigned strategy $\boldsymbol{r}_i(p_i)\in S_i$ irrespective of her opponent's monitoring ability $p_j\in P_j^*$.

The following theorem states that the Folk Theorem holds for every $p\in P^*$ with the above restrictions of limited knowledge.

Theorem 8: *For every feasible and individually rational payoff vector $v\in R^2$, every $(\boldsymbol{d}^m)_{m=1}^{\infty}$ satisfying $\lim_{m\to\infty}\boldsymbol{d}^m=1$, and every $\boldsymbol{e}>0$, there exists $(\boldsymbol{r}^m)_{m=1}^{\infty}$ such that for every $p\in P^*$ and every large enough m, $\boldsymbol{r}^m(p)$ is a Nash equilibrium in $\Gamma(\boldsymbol{d}^m)$, and*

$$v-(\boldsymbol{e},\boldsymbol{e})<\lim_{m\to\infty}v(\boldsymbol{d}^m,\boldsymbol{r}^m(p))<v+(\boldsymbol{e},\boldsymbol{e}).$$

Proof: Fix $(\boldsymbol{d}^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to\infty}\boldsymbol{d}^m=1$. From the compactness of P^*, there exist $\overline{e}>0$, $\underline{e}>0$, $\Psi_i^*\!:P_i^*\to 2^{\Omega_i}$ and $\Psi_i^{**}\!:P_i^*\to 2^{\Omega_i}$ for each $i=1,2$ such that

$$\overline{e}>\underline{e},$$

and for each $i=1,2$ and every $p_i\in P_i^*$,

$$\underline{e}=\int_{w_i\in\Psi_i^*(p_i)}p_i(w_i|c)dw_i=\int_{w_i\in\Psi_i^{**}(p_i)}p_i(w_i|d)dw_i,$$

and

$$\bar{e} = \int_{w_i \in \Psi_i^*(p_i)} p_i(w_i | c / d_j) dw_i = \int_{w_i \in \Psi_i^{**}(p_i)} p_i(w_i | d / c_j) dw_i .$$

For each $i = 1,2$ and every $p_i \in P_i^*$, We set the associated sets Ω_i^* and Ω_i^{**} introduced in Step 1 of the proof of Theorem 3 equivalent to $\Psi_i^*(p_i)$ and $\Psi_i^{**}(p_i)$, respectively. Hence, we can choose $(r_i^*(T))_{T=1}^{\infty}$, $(T^m)_{m=1}^{\infty}$ and $(\bar{v}^m, \underline{v}^m, (\bar{x}_i^m, \underline{x}_i^m)_{i=1,2})_{m=1}^{\infty}$ introduced in Step 1 of the proof of Theorem 3 independently of $p_i \in P_i^*$. We denote $\bar{s}_i^{m,p_i}$ and $\underline{s}_i^{m,p_i}$ instead of $\bar{s}_i^m$ and $\underline{s}_i^m$, respectively, which are the strategies specified in Step 1 of the Proof of Theorem 3. We specify $(\bar{r}_i^m)_{m=1}^{\infty}$ and $(\underline{r}_i^m)_{m=1}^{\infty}$ by

$$\bar{r}_i^m(p_i) = \bar{s}_i^{m,p_i} \text{ and } \underline{r}_i^m(p_i) = \underline{s}_i^{m,p_i} \text{ for all } p_i \in P_i^* .$$

Step 1 in the proof of Theorem 3 implies that for every $p \in P^*$ and every large enough m, $(\bar{r}_1^m(p_1), \bar{r}_2^m(p_2))$, $(\bar{r}_1^m(p_1), \underline{r}_2^m(p_2))$, $(\underline{r}_1^m(p_1), \bar{r}_2^m(p_2))$ and $(\underline{r}_1^m(p_1), r_2^m(p_2))$ are all Nash equilibria, approximately sustaining (1,1), (1,0), (0,1) and (0,0), respectively.

From the compactness of P^*, there exist $\bar{e}^+ > 0$, $\underline{e}^+ > 0$, and $\Psi_i^+ : P_i^* \to 2^{\Omega_i}$ for each $i = 1,2$ such that

$$\bar{e}^+ > \underline{e}^+ ,$$

and for each $i = 1,2$ and every $p_i \in P_i^*$,

$$\underline{e}^+ = \int_{w_i \in \Psi_i^+(p_i)} p_i(w_i | d / c_j) dw_i ,$$

and

$$\bar{e}^+ = \int_{w_i \in \Psi_i^+(p_i)} p_i(w_i | d) dw_i .$$

For every $i = 1,2$ and every $p_i \in P_i^*$, we set the associated set Φ_i^+ introduced in Step 2 of the proof of Theorem 3 equivalent to $\Psi_i^+(p_i)$. Hence, we can choose $(r_i^+(M))_{M=1}^{\infty}$ and $(M^m)_{m=1}^{\infty}$ introduced in Step 2 of the proof of Theorem 3 independently of $p_i \in P_i^*$. We denote $s_i^{[1,m,p_i]}$ and $s_i^{[2,m,p_i]}$ instead of $s_i^{[1,m]}$ and $s_i^{[2,m]}$, respectively, which are the strategies specified in Step 2 of the proof of Theorem 3. We specify $(r_i^{[1,m]})_{m=1}^{\infty}$ and $(r_i^{[2,m]})_{m=1}^{\infty}$ by

$$r_i^{[1,m]}(p_i) = s_i^{[1,m,p_i]} \text{ and } r_i^{[2,m]}(p_i) = s_i^{[2,m,p_i]} \text{ for all } p_i \in P_i^* .$$

Step 2 of the proof of Theorem 3 implies that for every $p \in P^*$ and every large enough m, both $(r_1^{[1,m]}(p_1), r_2^{[1,m]}(p_2))$ and $(r_1^{[2,m]}(p_1), r_2^{[2,m]}(p_2))$ are Nash equilibria, approximately sustaining $z^{[1]}$ and $z^{[2]}$, respectively.

Fix a positive real number $e > 0$, a positive integer K, and K feasible and individually rational payoff vectors $v^{\{1\}}, ..., v^{\{K\}}$, arbitrarily. Suppose that for every $k \in \{1, \ldots, K\}$, there exists $(r^{\{k,m\}})_{m=1}^{\infty}$ such that for every $p \in P^*$ and every large enough m, $r^{\{k,m\}}(p)$ is a Nash equilibrium in $\Gamma(d^m)$, and

$$v^{\{k\}} - (e,e) < \lim_{m\to\infty} v(d^m, r^{\{k,m\}}(p)) < v^{\{k\}} + (e,e).$$

We specify $(r^m)_{m=1}^{\infty}$ satisfying that for every $i \in \{1,2\}$, every $p_i \in P_i^*$, and every $k \in \{1,\dots,K\}$,

$$r_i^m(p_i)(h_i^{k-1}) = r_i^{\{k,m\}}(p_i)(h_i^0),$$

and for every $t \geq K+1$,

$$r_i^m(p_i)(h_i^{t-1}) = r_i^{\{k,m\}}(p_i)(\tilde{h}_i^{\tilde{t}})$$

if $t = K\tilde{t} + k$ and $(\tilde{a}_i(t), \tilde{w}_i(t)) = (a_i(Kt+k), w_i(Kt+k))$

for all $t = 1,\dots,\tilde{t}$.

Note that for every $p \in P^*$, $r^m(p)$ is a Nash equilibrium in $\Gamma((d^m)^{\frac{1}{K}})$ for every large enough $m = 1,2,\dots$, and

$$\lim_{m\to\infty} v((d^m)^{\frac{1}{K}}, r^m(p)) = \lim_{m\to\infty} \frac{\sum_{k=1}^{K} (d^m)^{\frac{k-1}{K}} v(d^m, r^{\{k,m\}}(p))}{\sum_{k=1}^{K} (d^m)^{\frac{k-1}{K}}}$$

$$\in [\frac{\sum_{k=1}^{K} v^{\{k\}}}{K} - (e,e), \frac{\sum_{k=1}^{K} v^{\{k\}}}{K} + (e,e)].$$

Hence, we have proved that $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ is sustainable.

Since the set of payoff vectors satisfying the conditions in Theorem 3 is compact, we have proved that every feasible and individually rational payoff vector satisfies the conditions of Theorem 3.

Q.E.D.

8.2. Uniform Sustainability

We reconsider uniform sustainability discussed in Section 7. We denote by P^{**} the set of all conditional density functions p satisfying $\bar{v}^{[p]} > \underline{v}^{[p]}$. For each $i = 1,2$, we define P_i^{**} as the set of all conditional density functions p_i on player $i's$ private signal satisfying that $p_i(w_i|a) \equiv \int_{w_j \in \Omega_j} p(w|a) dw_j$ for some $p \in P^{**}$. Note that players' private signals are not necessarily conditionally independent. We assume that each player i only knows which element of P_i^{**} is the correct conditional density function for her own private signal. We assume that it is common knowledge that the correct conditional density function belongs to P^{**}. We also assume that each player i has no idea on what is the degree of correlation between their private signals. A mapping assigning each element of P_i^{**} a rule for player i is denoted by $b_i : P_i^{**} \to \Sigma_i$. Let $b \equiv (b_1, b_2)$ and $b(p) \equiv (b_1(p_1), b_2(p_2))$. Player i behaves according to the assigned rule $b_i(p_i) \in \Sigma_i$ irrespective of her opponent's monitoring ability $p_j \in P_j^{**}$.

The following theorem states that for every $p \in P^{**}$, the Pareto-dominant uniformly sustainable payoff vector $\bar{v}^{[p]}$ can be uniformly sustained by a rule profile with the above restrictions of limited knowledge.

Theorem 9: *For every* $(d^m)_{m=1}^{\infty}$ *satisfying* $\lim_{m\to\infty} d^m = 1$ *and every* $e > 0$, *there exists* $(b^m)_{m=1}^{\infty}$ *such that for every* $p \in P^{**}$ *and every large enough* m, $b^m(p)$ *is a uniform equilibrium in* $\Gamma(d^m)$, *and for every* $q \in Q$

$$\bar{v}^{[p]} - (e,e) < \lim_{m\to\infty} v(d^m, b^m(p), q) < \bar{v}^{[p]} + (e,e).$$

Proof: Let $(s^m)_{m=1}^{\infty}$ be the infinite sequence of rule profiles defined in the proof of Theorem 6, where we assume $\bar{v}^{[p]} - (e,e) < v^- < v^+ < \bar{v}^{[p]} + (e,e)$. We will write $s^{m,p} = (s_1^{m,p_1}, s_2^{m,p_2})$ instead of s^m. Here, we must note that, by definition, s_i^m depends only on p_i for each $i = 1,2$. Hence, we can specify $(b^m)_{m=1}^{\infty}$ by

$$b_i^m(p_i) = s_i^{m,p_i} \text{ for each } i = 1,2 \text{, all } m = 1,2,\ldots \text{, and all } p \in P^{**}.$$

The proof of Theorem 6 implies that for every $p \in P^{**}$ and every large enough m, $b^m(p) = s^{m,p}$ is a uniform equilibrium in $\Gamma(d^m)$, and for every $q \in Q$,

$$\bar{v}^{[p]} - (e,e) < \lim_{m\to\infty} v(d^m, s^m, q) = \lim_{m\to\infty} v(d^m, b^m(p), q)$$
$$< \bar{v}^{[p]} + (e,e).$$

Q.E.D.

9. Conclusion and Future Research

The present paper investigated repeated prisoner-dilemma games with discounting where players are sufficiently patient. We provided the Folk Theorem in terms of Nash equilibrium when players' private signals are conditionally independent. We also showed that the zero likelihood ratio condition is necessary and sufficient for efficient uniform sustainability. These results hold true even if players have limited knowledge on their opponents' private signal structures.

We have the following problems to be solved in future research.

We have proved the Folk Theorem on the conditional independence assumption by using the review strategy equilibrium construction. The use of the review strategy relies on the conditional independence assumption. Hence, whether the Folk Theorem holds even without conditional independence is an open question. In the study of repeated games with public monitoring, Matsushima (1989) provided an idea of equilibrium construction of punishment and reward on hyperplanes. Subsequently, by using this idea, together with that of self-generation explored by Abreu, Pearce and Stacchetti (1990), Fudenberg, Levine and Maskin (1994) provided the Folk Theorem in the public monitoring case. In order to discover the Folk Theorem without the use of the review strategy, it would be a crucial step to apply the idea of punishment and reward on hyperplanes to the private monitoring case.

The present paper considered only repeated prisoner-dilemma games. It is important to clarify whether this paper can be extended to more general games. For example, we can extend Theorem 1 to a class of games with more than two actions in the following way. Suppose that a player i has an action d_i' other than actions c_i and d_i, and there exist $a \in [0,1]$ and $a' \in [0,1]$ such that

$$u_i(c/d_i') \leq a u_i(c) + (1-a) u_i(c/d_i),$$
$$p_j(\hat{w}_j|c/d_i') > a p_j(\hat{w}_j|c) + (1-a) p_j(\hat{w}_j|c/d_i),$$
$$u_i(d/d_i') \leq a' u_i(d) + (1-a') u_i(d/c_i),$$

and

$$p_j(\tilde{w}_j|d/d_i') < a' p_j(\tilde{w}_j|d) + (1-a') p_j(\tilde{w}_j|d/c_i),$$

where $\hat{w}_j$ and $\tilde{w}_j$ are the private signals for the opponent defined in the proof of Theorem 1. Since the choice of action d_i' is worse than a mixture of actions c_i and d_i, player i have no incentive to choose action d_i' when her opponent plays the strategy constructed in the proof of Theorem 1.

The study of private monitoring in general repeated games with more than two actions and more than two players, and also in general stochastic games, should be expected to be started in the near future.

References

Abreu, D., P. Milgrom, and D. Pearce (1991): "Information and Timing in Repeated Partnerships," *Econometrica* 59, 1713-1734.

Abreu, D., D. Pearce, and E. Stacchetti (1986): "Optimal Cartel Monitoring with Imperfect Information," *Journal of Economic Theory* 39, 251-269.

Abreu, D., D. Pearce, and E. Stacchetti (1990): " Towards a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica* 58, 1041-1064.

Abreu, D. and A. Rubinstein (1989): "The Structure of Nash Equilibrium in Repeated Games with Finite Automata," *Econometrica* 56, 1259-1281.

Bernheim, D. (1984): "Rationalizable Strategic Behavior," *Econometrica* 52, 1007-1027.

Bhaskar, V. (1999): "The Prisoner's Dilemma with Private Monitoring," mimeo.

Compte, O. (1998): "Communication in Repeated Games with Imperfect Monitoring," *Econometrica* 66, 597-626.

Ely, J. and J. Valimaki (1999): "A Robust Folk Theorem for the Prisoner's Dilemma," mimeo.

Fudenberg, D. and D. Levine (1991): "Approximate Equilibria in Repeated Games with Imperfect Private Information," *Journal of Economic Theory* 54, 26-47.

Fudenberg, D., D. Levine, and E. Maskin (1994): "The Folk Theorem with Imperfect Public Information," *Econometrica* 62, 997-1040.

Green, E. and R. Porter (1984): "Noncooperative Collusion under Imperfect Price Information," *Econometrica* 52, 87-100.

Kandori, M. and H. Matsushima (1998): "Private Observation, Communication and Collusion," *Econometrica* 66, 627-652.

Lehrer, E. (1989): "Lower Equilibrium Payoffs in Two-Player Repeated Games with Non-Observable Actions," *International Journal of Game Theory* 18, 57-89.

Matsushima, H. (1989): "Efficiency in Repeated Games with Imperfect Monitoring," *Journal of Economic Theory* 48, 428-442.

Matsushima, H. (1990a): "On the Theory of Repeated Games with Private Information: Part I," *Economics Letters* 35, 253-256.

Matsushima, H. (1990b): "On the Theory of Repeated Games with Private Information: Part II," *Economics Letters* 35, 257-261.

Matsushima, H. (1999): "Multimarket Contact, Imperfect Monitoring, and Implicit Collusion," forthcoming in *Journal of Economic Theory*.

Mailath, G. and S. Morris (1998): "Repeated Games with Imperfect Private Monitoring: Notes on a Coordination Perspective," mimeo.

Neyman, A. (1985): "Bounded Complexity Justifies Cooperation in the Finitely Repeated

Prisoners' Dilemma," *Economics Letters* 19, 227-229.

Pearce, D. (1984): "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica* 52, 1029-1050.

Piccione, M. (1998): "The Repeated Prisoners' Dilemma with Imperfect Private Monitoring," mimeo.

Radner, R. (1985): "Repeated Principal Agent Games with Discounting," *Econometrica* 53, 1173-1198.

Radner, R. (1986): "Repeated Partnership Games with Imperfect Monitoring and No Discounting," *Review of Economic Studies* 53, 43-47.

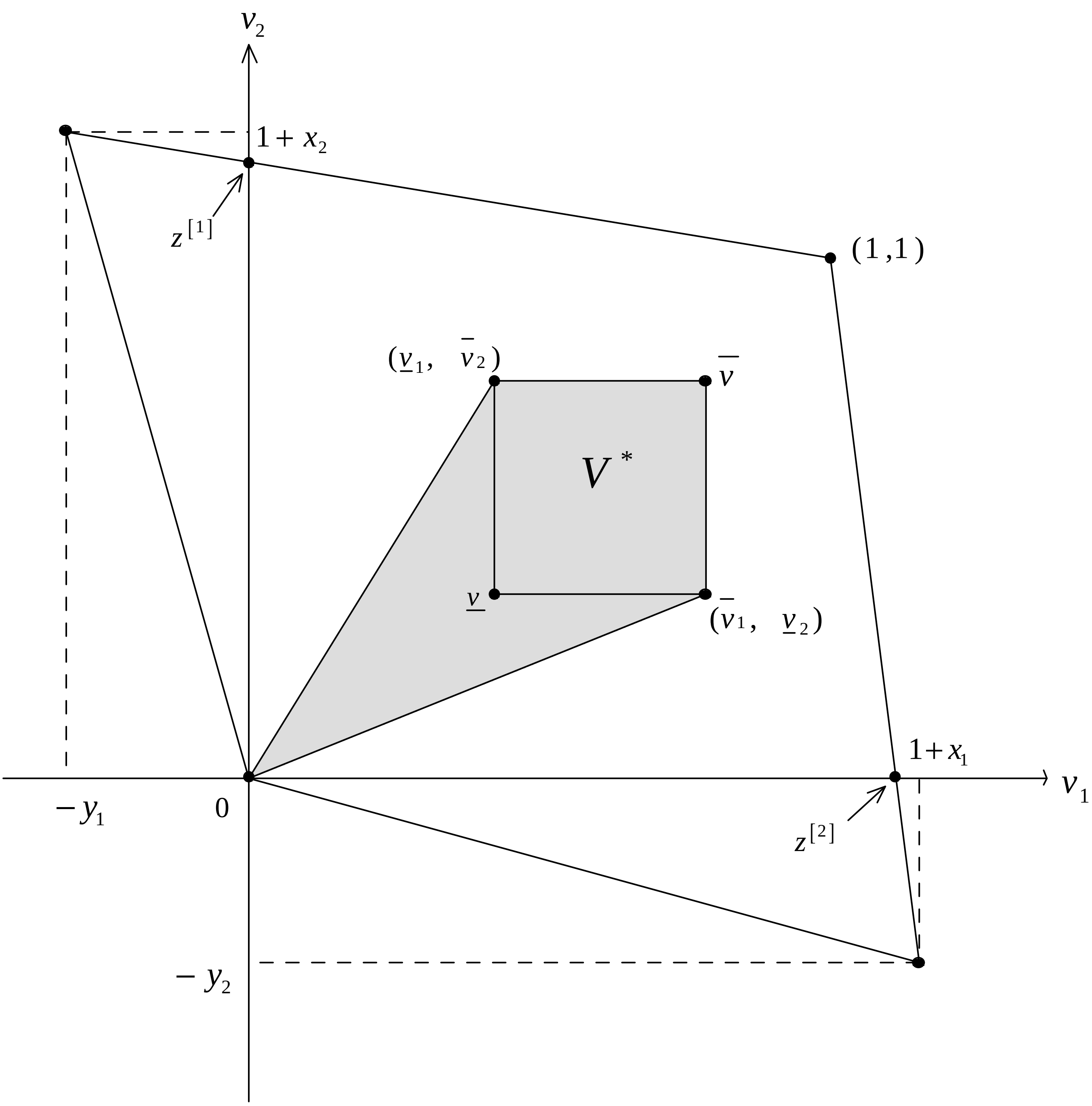
Rubinstein, A. (1986): "Finite Automata Play the Repeated Prisoner's Dilemma", *Journal of Economic Theory* 39, 83-96.

Scherer, F. and D. Ross (1990): *Industrial Market Structure and Economic Performance*, Boston: Houghton Mifflin Company.

Sekiguchi, T. (1997): "Efficiency in Repeated Prisoners' Dilemma with Private Monitoring," *Journal of Economic Theory* 76, 345-361.

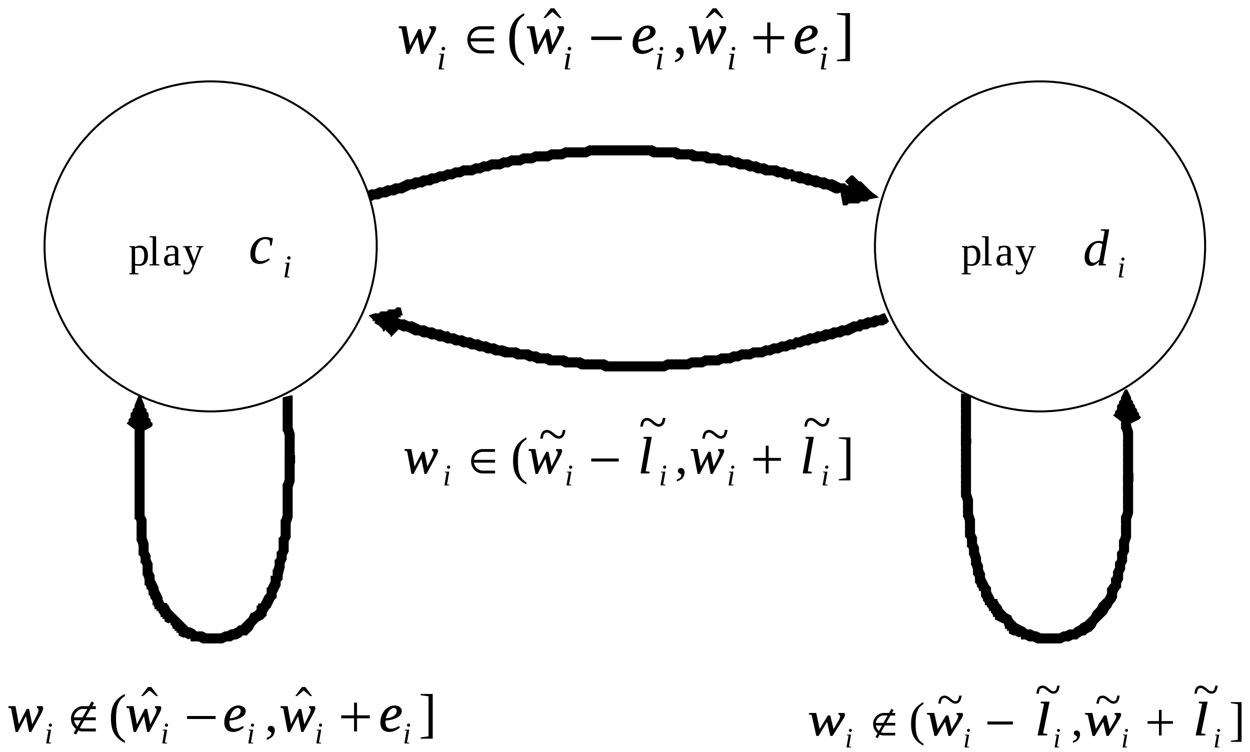
Stigler, G. (1964): "A Theory of Oligopoly," *Journal of Political Economy* 72, 44-61.

Tirole, J. (1988): *The Theory of Industrial Organization*, Cambridge: MIT Press.



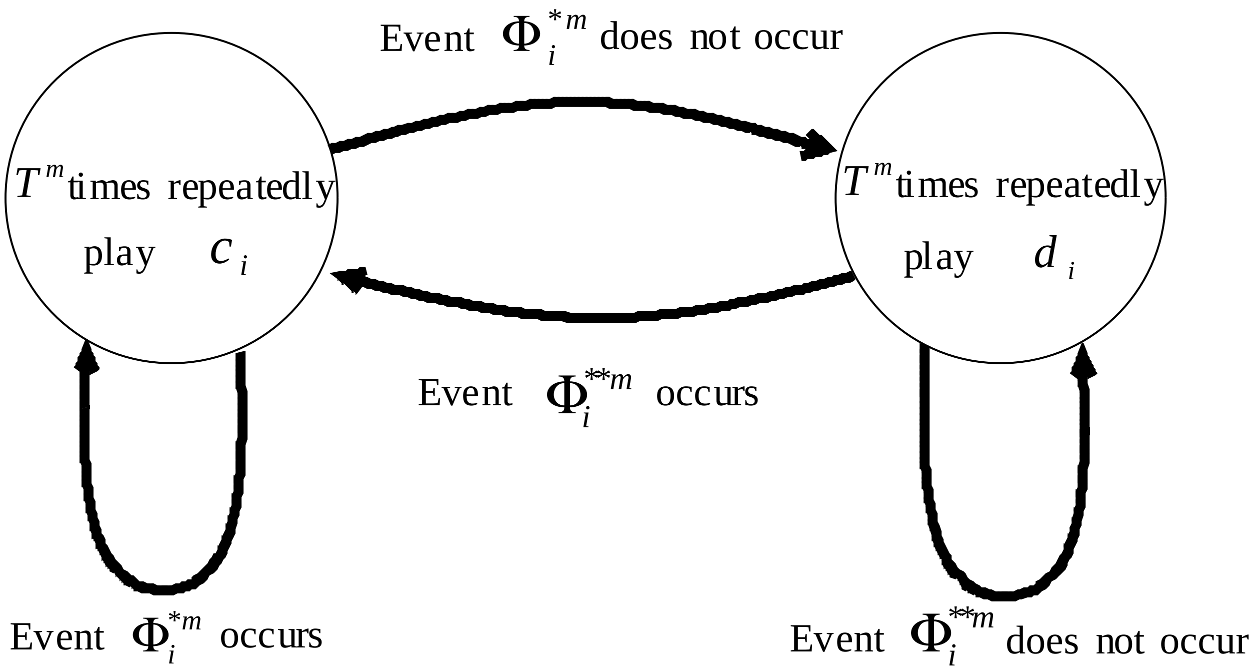
v_2
$1+x_2$
$z^{[1]}$
$(1,1)$
$(\underline{v}_1, \bar{v}_2)$
$\bar{v}$
V^*
$\underline{v}$
$(\bar{v}_1, \underline{v}_2)$
$1+x_1$
v_1
$-y_1$
0
$z^{[2]}$
$-y_2$

[Figure 1]

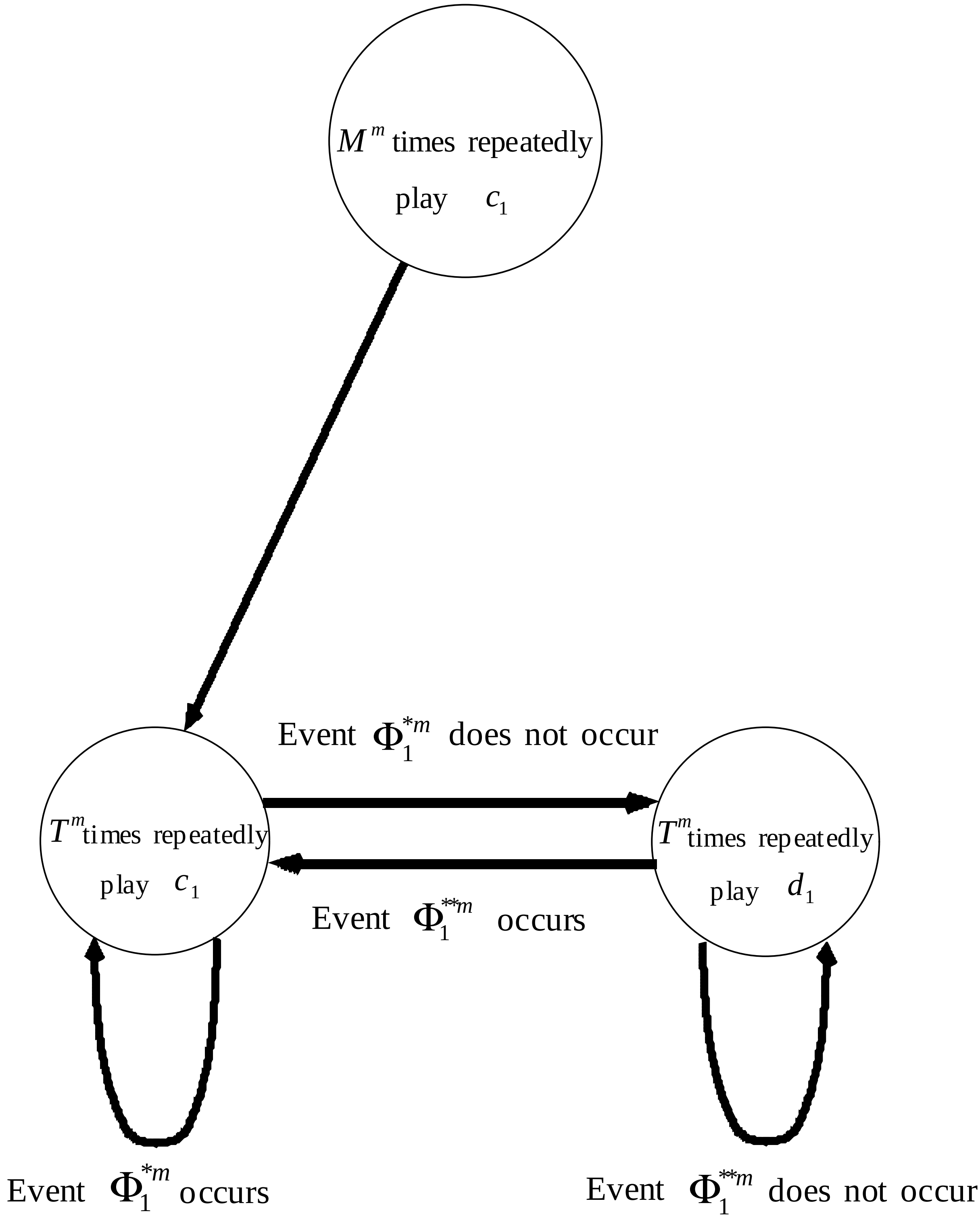


$w_i \in (\hat{w}_i - e_i, \hat{w}_i + e_i]$
play c_i
play d_i
$w_i \in (\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]$
$w_i \notin (\hat{w}_i - e_i, \hat{w}_i + e_i]$
$w_i \notin (\tilde{w}_i - \tilde{l}_i, \tilde{w}_i + \tilde{l}_i]$

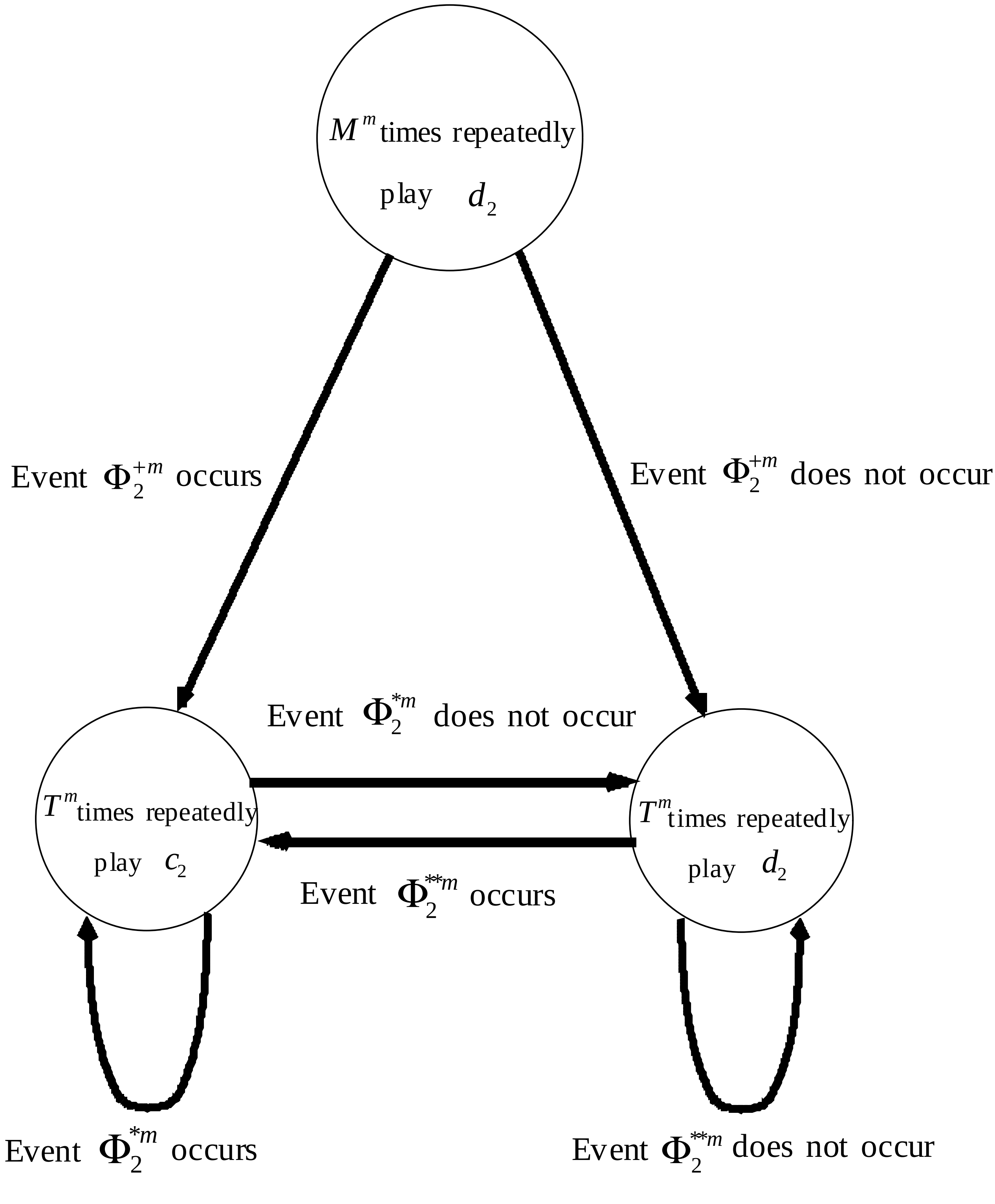
[Figure 2]


Event Φ_i^{*m} does not occur
T^m times repeatedly play c_i
T^m times repeatedly play d_i
Event Φ_i^{**m} occurs
Event Φ_i^{*m} occurs
Event Φ_i^{**m} does not occur

[Figure 3]


M^m times repeatedly play c_1
T^m times repeatedly play c_1
T^m times repeatedly play d_1
Event Φ_1^{*m} does not occur
Event Φ_1^{**m} occurs
Event Φ_1^{*m} occurs
Event Φ_1^{**m} does not occur

[Figure 4.1]



M^m times repeatedly play d_2
Event Φ_2^{+m} occurs
Event Φ_2^{+m} does not occur
Event Φ_2^{*m} does not occur
T^m times repeatedly play c_2
T^m times repeatedly play d_2
Event Φ_2^{**m} occurs
Event Φ_2^{*m} occurs
Event Φ_2^{**m} does not occur

[Figure 4.2]