CIRJE-F-77

Tail Probabilities of the Limiting Null Distributions of the Anderson-Stephens Statistics

Satoshi Kuriki
The Institute of Statistical Mathematics

Akimichi Takemura
University of Tokyo

June 2000

Tail probabilities of the limiting null distributions of the Anderson-Stephens statistics

Satoshi Kuriki
The Institute of Statistical Mathematics
and
Akimichi Takemura
Faculty of Economics, University of Tokyo

June, 2000

Abstract

For the purpose of testing the spherical uniformity based on i.i.d. directional data (unit vectors) z_i, $i = 1, \ldots, n$, Anderson and Stephens (1972) proposed testing procedures based on the statistics $S_{\max} = \max_u S(u)$ and $S_{\min} = \min_u S(u)$, where u is a unit vector and $nS(u)$ is the sum of square of $u'z_i$'s. In this paper we also consider another test statistic $S_{\text{range}} = S_{\max} - S_{\min}$. We provide formulas for the P-values of $S_{\max}$, $S_{\min}$, S_{range} by approximating tail probabilities of the limiting null distributions by means of the tube method, an integral-geometric approach for evaluating tail probability of the maximum of a Gaussian random field. Monte Carlo simulations for examining the accuracy of the approximation and for the power comparison of the statistics are given.

Key words: directional data, integral geometry, maximum of a Gaussian field, multivariate symmetric normal distribution, test for spherical uniformity, Weyl's tube formula.

1 Introduction

Assume that q-dimensional i.i.d. directional data (unit column vectors) z_i, $i = 1, \ldots, n$, are observed. Consider the hypothesis that z_i has the uniform distribution on the unit sphere S^{q-1} in R^q. For testing this null hypothesis of spherical uniformity, Anderson and Stephens (1972) proposed testing procedures with critical regions

$$S_{\max} = \max_{u \in S^{q-1}} S(u) > c \quad \text{or} \quad S_{\min} = \min_{u \in S^{q-1}} S(u) < c',$$

where

$$nS(u) = \sum_{i=1}^{n} (u'z_i)^2, \quad u \in S^{q-1},$$

is the sum of square of the components of z_i's with respect to the direction u. Obviously the test statistics $S_{\max}$ and $S_{\min}$ are the largest and smallest eigenvalues $\lambda_1(Q)$ and $\lambda_q(Q)$ of a $q \times q$ matrix $Q = (1/n)\sum_{i=1}^{n} z_i z_i'$, respectively. Under the null hypothesis the matrix Q has expectation $(1/q)I_q$, and the eigenvalues of Q far away from the value $1/q$ indicates departure from the null hypothesis. Anderson and Stephens (1972) considered two types of alternatives, the bimodal and equatorial alternatives, where the data z_i's are concentrated or deconcentrated with respect to a particular axis, and proposed the test statistics $S_{\max}$ and $S_{\min}$. In this paper we propose another test procedure with a critical region

$$S_{\text{range}} = \max_{u,v \in S^{q-1}} (S(u) - S(v)) = S_{\max} - S_{\min} > c'',$$

which is expected to detect different types of alternatives than the original Anderson-Stephens statistics. In the succeeding section, we will examine the power performances of the Anderson-Stephens statistics and their modification S_{range}. The motivation for S_{range} shall be made clearer there.

In order to give critical points for $S_{\max}$, $S_{\min}$ and S_{range}, we consider the limiting distributions when the sample size n goes to infinity. The limiting null distribution of any subset of the eigenvalues of $\sqrt{n}(Q - (1/q)I_q)$ is given by the corresponding marginal distribution of the joint density (2) in page 617 of Anderson and Stephens (1972) (see also Section 2.3 of Watson (1983)). The density given there is easily shown to be the joint density of the eigenvalues of

$$\sqrt{\frac{2}{q(q+2)}}\Big(A - \frac{\text{tr}(A)}{q} I_q\Big),$$

where $A = (a_{ij})$ is a $q \times q$ symmetric random matrix whose diagonal elements a_{ii} and upper off-diagonal elements a_{ij} $(i < j)$ are independently distributed as $a_{ii} \sim N(0,1)$, the standard normal distribution, and $a_{ij} \sim N(0, 1/2)$, respectively. The distribution of A is sometimes called $(q \times q)$ *multivariate symmetric normal distribution* (e.g., Siotani *et al.* (1985), page 159). The lemma below follows immediately from this fact.

Lemma 1.1 *As $n \to \infty$, the null distributions of both of $\sqrt{n}(S_{\max} - 1/q)$ and $-\sqrt{n}(S_{\min} - 1/q)$ converge to the distribution of $\sqrt{2(q-1)/q^2(q+2)}\, T_1$, where*

$$T_1 = \lambda_1(B) \quad \textit{with} \quad B = \sqrt{\frac{q}{q-1}}\Big(A - \frac{\text{tr}(A)}{q} I_q\Big). \tag{1}$$

The null distribution of $\sqrt{n}(S_{\max} - S_{\min})$ converges to the distribution of $(2/\sqrt{q(q+2)})\, T_2$, where

$$T_2 = \frac{1}{\sqrt{2}}(\lambda_1(A) - \lambda_q(A)). \tag{2}$$

The purpose of this paper is to provide approximate formulas for upper tail probabilities $P(T_1 \geq x)$ and $P(T_2 \geq x)$ in the form of valid asymptotic expansions as $x \to \infty$. The obtained formulas are shown to be sufficiently accurate for calculating P-values. In order to derive the formulas, we take the *tube method*, an integral-geometric approach originating from Hotelling (1939) and Weyl (1939). Sun (1993) showed that an approximate tail probability formula for the maximum of a Gaussian random field with a constant variance can be obtained via the tube formula of Hotelling (1939) and Weyl (1939). The upper and lower bounds for the approximate formula by the tube method are given by Kuriki and Takemura (1998). Applications of the tube method to multivariate analysis are found in Sun (1991), Park and Sun (1998), and Kuriki and Takemura (1998). See also Knowles and Siegmund (1989), Naiman (1990), and the references therein.

The outline of this paper is as follows. In Section 2, we first explain that the statistics $T_1 = \lambda_1(B)$ and $T_2 = (\lambda_1(A) - \lambda_q(A))/\sqrt{2}$ can be reduced to canonical forms which can be dealt with by the tube method, and give the tail probability formulas for the statistics in Theorems 2.1 and 2.2. Furthermore we present numerical examples for confirming the accuracy of the obtained formulas and for power comparisons of the test statistics. Proofs of the theorems are given in Section 3. A summary of the tube method from Kuriki and Takemura (1998) is given in Appendix A.1. The rest of the Appendix is devoted to some mathematical details which are required in the proof of Theorem 2.2. In particular we explicitly evaluate the moment $E[\det(A)^2]$ of a multivariate symmetric normal matrix A (see Lemma A.4), which might be of some independent interest.

2 Main results

2.1 Tail probabilities of the statistics

Let $\mathrm{Sym}(q)$ denote the vector space of $q \times q$ real symmetric matrices endowed with the inner product $\langle X, Y \rangle = \mathrm{tr}(XY)$, $X, Y \in \mathrm{Sym}(q)$. $\mathrm{Sym}(q)$ can be identified with $R^{q(q+1)/2}$ with the usual Euclidean norm by identifying an element $X = (x_{ij}) \in \mathrm{Sym}(q)$, $x_{ij} = z_{ii}$ $(i = j)$, $z_{ij}/\sqrt{2}$ $(i < j)$, $z_{ji}/\sqrt{2}$ $(i > j)$, with $(z_{11}, \ldots, z_{qq}, z_{12}, z_{13}, \ldots, z_{q-1,q}) \in R^{q(q+1)/2}$. Note that the $q \times q$ multivariate symmetric normal distribution corresponds to the $q(q+1)/2$-dimensional multivariate standard normal distribution $N_{q(q+1)/2}(0, I_{q(q+1)/2})$.

Consider two submanifolds of $\mathrm{Sym}(q)$,

$$M_1 = \left\{ \sqrt{\frac{q}{q-1}}(uu' - (1/q)I_q) \mid u \in S^{q-1} \right\}$$

and

$$M_2 = \left\{ \frac{1}{\sqrt{2}}(uu' - vv') \mid u, v \in S^{q-1},\ u'v = 0 \right\}.$$

It is easy to see that the manifolds M_1 and M_2 are submanifolds of the unit sphere in $\mathrm{Sym}(q)$,

$$S^{q(q+1)/2-1} = \{X \in \mathrm{Sym}(q) \mid \mathrm{tr}(X^2) = 1\}.$$

Also we can see that

$$T_1 = \lambda_1(B) = \max_{U \in M_1} \mathrm{tr}(UA)$$

and

$$T_2 = \frac{1}{\sqrt{2}}(\lambda_1(A) - \lambda_p(A)) = \max_{U \in M_2} \mathrm{tr}(UA),$$

where A is a $q \times q$ matrix distributed as the multivariate symmetric normal distribution, and B is a symmetric $q \times q$ random matrix defined in (1). Now T_1 and T_2 are expressed in canonical forms and the upper probabilities $P(T_1 \geq x)$ and $P(T_2 \geq x)$ can be evaluated by the tube method in the form of valid asymptotic expansions as $x \to \infty$ (see (27) of Appendix A.1).

We summarize the main results of this paper as Theorems 2.1 and 2.2. The proofs of the theorems are given in Section 3. The upper probability of the χ^2 distribution with m degrees of freedom is denoted by $\bar{G}_m(\cdot)$.

Theorem 2.1 *When $q \geq 3$, the asymptotic expansion of the upper tail probability of $T_1 = \lambda_1(B)$ is given by*

$$P(T_1 \geq x) = \sum_{e=0,\, e:\mathrm{even}}^{q-1} w_{q-e}\, \bar{G}_{q-e}(x^2) + O\Big(\bar{G}_{q(q+1)/2-1}\Big(\frac{2q-2}{q-2}x^2\Big)\Big), \quad x \to \infty, \tag{3}$$

where

$$w_{q-e} = \frac{1}{2}\Big(\frac{2q}{q-1}\Big)^{(q-1)/2}\Big(-\frac{q+1}{2q}\Big)^{e/2}\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q-e+1}{2})\,(\frac{e}{2})!}. \tag{4}$$

When $q = 2$,

$$P(T_1 \geq x) = \bar{G}_2(x^2), \quad x \geq 0.$$

Remark 2.1 *When q is odd, it holds that $2\sum_{i:\,odd} w_i = 1$. This is a consequence of the Gauss-Bonnet theorem and the fact that the Euler characteristic of the index set M_1 for q odd is 1. (See, e.g., Takemura and Kuriki (1999), Corollary 3.1.)*

Theorem 2.2 *When $q \geq 3$, the asymptotic expansion of the upper tail probability of $T_2 = (\lambda_1(A) - \lambda_q(A))/\sqrt{2}$ is given by*

$$P(T_2 \geq x) = \sum_{e=0,\, e:\mathrm{even}}^{2q-3} w_{2q-2-e}\, \bar{G}_{2q-2-e}(x^2) + O(\bar{G}_{q(q+1)/2-1}(4x^2/3)), \quad x \to \infty, \tag{5}$$

where

$$w_{2q-2-e} = 2^{q-2}\Big(-\frac{1}{2}\Big)^{e/2}\binom{q}{e/2}. \tag{6}$$

When $q = 2$,

$$P(T_2 \geq x) = \bar{G}_2(x^2), \quad x \geq 0.$$

Remark 2.2 *Upper and lower bounds for $P(T_1 \geq x)$ and $P(T_2 \geq x)$ can be given by Theorem 3.1 of Kuriki and Takemura (1998).*

2.2 Numerical examples

2.2.1 Null distributions with finite/infinite sample sizes

Consider the statistics T_1, T_2 in (1), (2) for $q = 3$. The approximation for T_1 by Theorem 2.1 is

$$P(T_1 \ge x) \sim \frac{3}{2}\bar{G}_3(x^2) - \bar{G}_1(x^2),$$

whereas the exact probability given in page 617 of Anderson and Stephens (1972) is

$$P(T_1 \ge x) = \frac{3}{2}\bar{G}_3(x^2) - \bar{G}_1(x^2) + \frac{1}{2}\bar{G}_1(4x^2), \quad x \ge 0. \tag{7}$$

Note that the difference $\bar{G}_1(4x^2)/2$ is within the order of $O(\bar{G}_5(4x^2))$ given in Theorem 2.1.

The approximation for T_2 by Theorem 2.2 is

$$P(T_2 \ge x) \sim 2\bar{G}_4(x^2) - 3\bar{G}_2(x^2),$$

whereas the exact probability can be evaluated as

$$\begin{aligned} P(T_2 \ge x) &= 2\bar{G}_4(x^2) - 3\bar{G}_2(x^2) \\ &\quad - \int_x^\infty (y^3 - 3y)\bar{G}_1(y^2/3)\, e^{-y^2/2}\, dy + \frac{9}{8}\bar{G}_3(4x^2/3), \quad x \ge 0. \end{aligned} \tag{8}$$

In Figures 2.1 (or 2.2) and 2.3, the approximate and the exact tail probabilities of T_1 and T_2 are are plotted. We see that the asymptotic expansion by the tube method give very satisfactory approximation to the limiting distribution.

Moreover, in order to examine the convergence speed as the sample size n goes to infinity, we plot the upper probability curves for $\sqrt{45n/4}\,(S_{\max} - 1/3)$, $-\sqrt{45n/4}\,(S_{\min} - 1/3)$ and $\sqrt{15n/4}\,S_{\mathrm{range}}$ estimated by Monte Carlo simulations with 50,000 replications in Figures 2.1–2.3. In each figure we see that the curve for $n = 100$ is close to that for $n = \infty$, and the curve for $n = 1000$ is almost indistinguishable from that for $n = \infty$.

2.2.2 Asymptotic power comparisons

In order to characterize the three statistics $S_{\max}$, $S_{\min}$ and S_{range}, we compare their asymptotic powers. We assume that n i.i.d. directional data z_i are obtained by normalizing the n Gaussian random vectors, i.e.,

$$z_i = x_i/\|x_i\|, \quad x_i \sim N_q(0, \Sigma), \quad i = 1, \ldots, n,$$

and consider the null hypothesis $\Sigma = kI_q$ for some $k > 0$ against a contiguous alternative hypothesis

$$\Sigma = k\Big(I_q + \sqrt{\frac{2(q+2)}{qn}}\,\Delta\Big) \quad \text{for some } k > 0,$$

where Δ is a $q \times q$ symmetric matrix. Under this local alternative, the limiting powers of $S_{\max}$, $S_{\min}$ and S_{range} are given by

$$P_\Delta(T_1 \ge c_1(\alpha)), \quad P_{-\Delta}(T_1 \ge c_1(\alpha)) \quad \text{and} \quad P_\Delta(T_2 \ge c_2(\alpha)),$$

where $P_\Delta(\cdot)$ means that the symmetric random matrix $A = (a_{ij})$ in T_1 and T_2 is distributed as the multivariate symmetric normal distribution with the expectation $E[A] = \Delta = (\delta_{ij})$, that is, the diagonal elements and the upper off-diagonal elements a_{ii} and a_{ij} $(i < j)$ are independently distributed as $a_{ii} \sim N(\delta_{ii}, 1)$ and $a_{ij} \sim N(\delta_{ij}, 1/2)$. $c_1(\alpha)$ and $c_2(\alpha)$ are $100\alpha\%$ critical points of T_1 and T_2.

The results for $q = 3$ are summarized in Table 2.1. Without loss of generality we restrict our attention to the case where Δ is diagonal and $\mathrm{tr}(\Delta) = 0$. We consider three cases, where Δ is proportional to $\Delta_1 = \mathrm{diag}(2, -1, -1)/\sqrt{6}$ (bimodal alternative), $-\Delta_1$ (equatorial alternative), and $\Delta_2 = \mathrm{diag}(1, 0, -1)/\sqrt{2}$. The critical points are obtained by the exact tail probability formulas (7) and (8). However in this table we omit the case $\Delta = -\Delta_1$ since the asymptotic powers of $S_{\max}$, $S_{\min}$, S_{range} for $\Delta = -\Delta_1$ are equivalent to those of $S_{\min}$, $S_{\max}$, S_{range} for $\Delta = \Delta_1$, respectively. Note also that when $\Delta = \Delta_2$, $S_{\max}$ and $S_{\min}$ give the same asymptotic powers.

From Table 2.1 we see that the power performance of the statistic $S_{\max}$ (or $S_{\min}$) is superior when $\Delta = \Delta_1$ (or $-\Delta_1$), where one eigenvalue of Δ is outstandingly large (or small, resp.). The performance of the statistic S_{range} is superior when $\Delta = \Delta_2$, where there exist positive and negative eigenvalues of Δ with large absolute values. Also S_{range} has moderate local powers even for $\Delta = \Delta_1$ and $-\Delta_1$.

3 Proofs by the tube method

We give proofs of Theorems 2.1 and 2.2 in Sections 3.1 and 3.2, respectively. Each proof consists of three parts. First, the geometric quantities of the index set such as the volume element and the second fundamental form are determined. Second, the coefficients w_{d+1-e} in the tube formula are derived. Finally, the critical radius θ_c of the index set which determines the remainder term of the asymptotic expansion is obtained.

3.1 The proof of Theorem 2.1

3.1.1 Geometry of the manifold M_1

Let $t = (t^1, \dots, t^{q-1})'$ be a local coordinate system of S^{q-1} so that $h \in S^{q-1}$ has a representation $h = h(t)$. Then $\phi \in M_1$ is written as

$$\phi = \phi(t) = \sqrt{\frac{q}{q-1}}(h(t)h(t)' - (1/q)I_q).$$

The dimension of M_1 is $d = \dim(M_1) = q - 1$. Note that M_1 is degenerate in the sense that M_1 is contained in a subspace

$$\{X \in \mathrm{Sym}(q) \mid \mathrm{tr}(X) = 1\}. \tag{9}$$

Indeed (9) is shown to be the linear hull of M_1 of dimension $p' = q(q+1)/2 - 1$.

Derivative with respect to t^i is denoted by the subscript i. For example, $h_i = \partial h/\partial t^i$, $\phi_i = \partial \phi/\partial t^i$, $\phi_{ij} = \partial^2 \phi/\partial t^i \partial t^j$. The tangent space $T_\phi(M_1)$ of M_1 in $\mathrm{Sym}(q)$ at $\phi = \phi(t)$ is spanned by

$$\phi_i = \sqrt{\frac{q}{q-1}}(h_i h' + h h_i'), \quad i = 1, \ldots, q-1. \tag{10}$$

Note that $h_i' h = 0$ since $h'h = 1$. The metric tensor at ϕ is

$$g_{ij} = \phi_i' \phi_j = \frac{2q}{q-1} h_i' h_j, \quad i, j = 1, \ldots, q-1. \tag{11}$$

Let dh and $d\phi$ denote the volume elements of S^{q-1} and M_1, respectively. Since $dh = \det(h_i' h_j)^{1/2} \prod_{i=1}^{q-1} dt^i$,

$$d\phi = \det(g_{ij})^{1/2} \prod_{i=1}^{q-1} dt^i = \left(\frac{2q}{q-1}\right)^{(q-1)/2} dh.$$

Noting that the multiplicity of the map $h \mapsto \phi = \sqrt{q/(q-1)}(hh' - (1/q)I_q)$ is 2, we have the following.

Lemma 3.1 *The total volume of M_1 is*

$$\mathrm{Vol}(M_1) = \left(\frac{2q}{q-1}\right)^{(q-1)/2} \Omega_q \times \frac{1}{2} = \left(\frac{2q}{q-1}\right)^{(q-1)/2} \frac{\pi^{q/2}}{\Gamma(q/2)},$$

where

$$\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$$

is the volume of the unit sphere S^{q-1}.

Let H be a $q \times (q-1)$ matrix such that (h, H) is orthogonal. Then $\phi \in M_q$ is written as

$$\phi = \begin{pmatrix} h & H \end{pmatrix} \begin{pmatrix} \sqrt{\frac{q-1}{q}} & 0 \\ 0 & -\frac{1}{\sqrt{q(q-1)}} I_{q-1} \end{pmatrix} \begin{pmatrix} h' \\ H' \end{pmatrix}.$$

The basis (10) of the tangent space $T_\phi(M_1)$ is written as

$$\phi_i = \sqrt{\frac{q}{q-1}} \begin{pmatrix} h & H \end{pmatrix} \begin{pmatrix} 0 & h_i' H \\ H' h_i & 0 \end{pmatrix} \begin{pmatrix} h' \\ H' \end{pmatrix}, \quad i = 1, \ldots, q-1.$$

Therefore, it is easy to verify that the orthogonal complement space $(\mathrm{span}\{\phi\} \oplus T_\phi(M_1))^\perp$ in $\mathrm{Sym}(q)$ is spanned by

$$\nu = \begin{pmatrix} h & H \end{pmatrix} \begin{pmatrix} \frac{1}{q-1}\mathrm{tr}(A) & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} h' \\ H' \end{pmatrix}, \quad A \in \mathrm{Sym}(q-1). \tag{12}$$

Note that

$$\mathrm{tr}(\nu^2) = \frac{1}{(q-1)^2}\mathrm{tr}(A)^2 + \mathrm{tr}(A^2). \tag{13}$$

The inner product of ν and a second derivative

$$\phi_{ij} = \sqrt{\frac{q}{q-1}}(h_{ij}h' + hh'_{ij} + h_i h'_j + h_j h'_i)$$

of ϕ is

$$\begin{aligned} \mathrm{tr}(\nu\phi_{ij}) &= \sqrt{\frac{q}{q-1}}\Big(2h'_{ij}h\frac{1}{q-1}\mathrm{tr}(A) + 2h'_i HAH'h_j\Big) \\ &= 2\sqrt{\frac{q}{q-1}}h'_i H\Big(A - \frac{1}{q-1}\mathrm{tr}(A)I_{q-1}\Big)H'h_j. \end{aligned}$$

Recalling that the metric is given by (11), we have the following lemma.

Lemma 3.2 *In an appropriate coordinate system, the second fundamental form of M_1 at ϕ with respect to the direction ν in (12) can be written as*

$$H(\phi,\nu) = -\sqrt{\frac{q-1}{q}}\Big(A - \frac{1}{q-1}\mathrm{tr}(A)I_{q-1}\Big). \tag{14}$$

3.1.2 The coefficients in the tube formula

We now proceed to evaluation of the coefficients w_{q-e} in (4). For fixed $\phi \in M_1$ we first evaluate the expectation

$$E[\mathrm{tr}_e H(\phi, N)] \tag{15}$$

in (30) of Appendix A.1, where $N \in \mathrm{Sym}(q)$ has the standard normal distribution in the linear subspace $(\mathrm{span}\{\phi\} \oplus T_\phi(M_1))^\perp$.

Let $\mathbf{1}_{q-1}$ or $\mathbf{1}$ be a $(q-1)\times 1$ vector consisting of 1. Assume that A in (12) is a symmetric normal random matrix whose upper off-diagonal elements a_{ij} $(i<j)$ are independently distributed as $N(0,1/2)$ and the vector of diagonal elements $(a_{11},\dots,a_{q-1,q-1})'$ is distributed as $N_{q-1}(0, I_{q-1} - (1/q(q-1))\mathbf{1}\mathbf{1}')$, independently of a_{ij} $(i<j)$. Then it is easily shown that $\mathrm{tr}(\nu^2)$ in (13) has the χ^2 distribution with $(q-1)q/2$ degrees of freedom. This implies that the distribution of (12) is the multivariate standard normal distribution in the space $(\mathrm{span}\{\phi\} \oplus T_\phi(M_1))^\perp$. On the other hand, the second fundamental form in (14) is rewritten as

$$H = H(\phi,\nu) = -\sqrt{\frac{q-1}{q}}(\mathrm{diag}(\bar{b}) + \bar{A}),$$

where $\bar{A} = (\bar{a}_{ij})$ with $\bar{a}_{ij} = 0$ $(i = j)$, a_{ij} $(i \neq j)$, and

$$\bar{b} = (\bar{b}_1, \ldots, \bar{b}_{q-1})' = (I_{q-1} - (q-1)^{-1}\mathbf{1}\mathbf{1}')(a_{11}, \ldots, a_{q-1,q-1})'.$$

Note that $\bar{b} \sim N_{q-1}(0, I_{q-1} - (q-1)^{-1}\mathbf{1}\mathbf{1}')$.

Lemma 3.3

$$E[\mathrm{tr}_e H] = \begin{cases} \binom{q-1}{e}\Big(-\dfrac{q+1}{2q}\Big)^{e/2}(e-1)!! & \text{for } e \text{ even}, \\ 0 & \text{for } e \text{ odd}, \end{cases}$$

where $(e-1)!! = (e-1)(e-3)\cdots 3 \cdot 1$.

Proof. Note first that the generalized trace $\mathrm{tr}_e H$ of H can be written as

$$\mathrm{tr}_e H = \sum_{|I|=e} \det H[I],$$

where $H[I]$ with $I = \{1 \leq i_1 < \cdots < i_e \leq q-1\}$ denotes the $e \times e$ submatrix of H formed by deleting all but columns and rows of H numbered $i_1, \ldots, i_e$ (Muirhead (1982), Appendix A7). Therefore

$$E[\mathrm{tr}_e H] = \binom{q-1}{e} E[\det H_e], \tag{16}$$

where $H_e = \mathrm{diag}(\bar{b}_1, \ldots, \bar{b}_e) + \bar{A}_e$ with $\mathrm{diag}(\bar{b}_1, \ldots, \bar{b}_e)' \sim N_e(0, I_e - (q-1)^{-1}\mathbf{1}_e\mathbf{1}_e')$, $\bar{A}_e = (\bar{a}_{ij})$ such that $\bar{a}_{ii} = 0$, $\bar{a}_{ij} = \bar{a}_{ji} \sim N(0, 1/2)$, $i < j$. Moreover

$$E[\det H_e] = E[\det(\mathrm{diag}(\bar{b}_1, \ldots, \bar{b}_e) + \bar{A}_e)] = \sum_{f=0}^{e} \binom{e}{f} E[\bar{b}_1 \cdots \bar{b}_f]\, E[\det \bar{A}_{e-f}]. \tag{17}$$

Since $E[\bar{b}_i] = 0$,

$$E[\bar{b}_1 \cdots \bar{b}_f] = \sum \mathrm{cov}(\bar{b}_{i_1}, \bar{b}_{i_2}) \cdots \mathrm{cov}(\bar{b}_{i_{f-1}}, \bar{b}_{i_f})$$

for f even, where the summation is taken over the set of all pairings $\{(i_1, i_2), \ldots, (i_{f-1}, i_f)\}$ of $\{1, \ldots, f\}$. Therefore

$$E[\bar{b}_1 \cdots \bar{b}_f] = \begin{cases} \mathrm{cov}(\bar{b}_1, \bar{b}_2)^{f/2}(f-1)!! = (-1/(q-1))^{f/2}(f-1)!! & \text{for } f \text{ even}, \\ 0 & \text{for } e \text{ odd}. \end{cases} \tag{18}$$

Also by expanding the determinant and taking the termwise expectation, we have

$$E[\det \bar{A}_{e-f}] = \begin{cases} (-1/2)^{(e-f)/2}\,(e-f-1)!! & \text{for } e-f \text{ even}, \\ 0 & \text{for } e-f \text{ odd}. \end{cases} \tag{19}$$

Combining (16)–(19), we have proven the lemma. ∎

As we have just seen, the expectation (15) does not depend on ϕ. Therefore the integration in (30) with respect to $d\phi$ over M_1 is reduced to multiplication by the constant $\mathrm{Vol}(M_1)$. Then from (30) the coefficient of the tube formula (29) for M_1 is

$$w_{q-e} = \frac{\Gamma(\frac{q-e}{2})}{2^{e/2+1}\,\pi^{q/2}}\,\mathrm{Vol}(M_1) \cdot E[\mathrm{tr}_e H],$$

which is reduced to (4) in Theorem 2.1.

3.1.3 Critical radius of the manifold M_1

We obtain the critical radius θ_c of the manifold M_1, which determines the order of the remainder term in (3).

Let $\phi = \sqrt{q/(q-1)}(hh' - (1/q)I_q)$ be a point of M_1. ϕ_i, $i = 1, \dots, q-1$, in (10) form a basis of $T_\phi(M_1)$. The orthogonal projection of $\tilde{\phi} \in M_1$ onto $\mathrm{span}\{\phi\} \oplus T_\phi(M_1)$ is given by

$$P_\phi(\tilde{\phi}) = \phi\,\mathrm{tr}(\phi\tilde{\phi}) + \sum_{i,j=1}^{q-1} \phi_i g^{ij} \mathrm{tr}(\phi_j \tilde{\phi}),$$

where g^{ij} is the (i,j)-th element of the inverse of the metric (g_{ij}) in (11). For $\tilde{\phi} = \sqrt{q/(q-1)}(\tilde{h}\tilde{h}' - (1/q)I_q) \neq \phi$, we have $\mathrm{tr}(\phi\tilde{\phi}) = (q/(q-1))(\tilde{h}'h - 1/q)$, $\mathrm{tr}(\phi_i\tilde{\phi}) = (2q/(q-1))(\tilde{h}'h)(\tilde{h}'h_i)$, and

$$\begin{aligned}
\mathrm{tr}(\tilde{\phi}P_\phi(\tilde{\phi})) &= \mathrm{tr}(\phi\tilde{\phi})^2 + \sum_{i=1}^{q-1} \mathrm{tr}(\phi_i\tilde{\phi}) g^{ij} \mathrm{tr}(\phi_j\tilde{\phi}) \\
&= \Bigl(\frac{q}{q-1}\Bigr)^2 \Bigl((\tilde{h}'h)^2 - \frac{1}{q}\Bigr)^2 + \Bigl(\frac{2q}{q-1}\Bigr)(\tilde{h}'h)^2 \tilde{h}'HH'\tilde{h} \\
&= \Bigl(\frac{q}{q-1}\Bigr)^2 \Bigl(-\frac{q-2}{q}x^4 + \frac{2(q-2)}{q}x^2 + \frac{1}{q^2}\Bigr),
\end{aligned}$$

where $x = \tilde{h}'h$. By virtue of Lemma A.1,

$$\begin{aligned}
\cot^2\theta_c &= \sup_{\tilde{\phi},\phi\in M_1} \frac{1 - \mathrm{tr}(\tilde{\phi}P_\phi(\tilde{\phi}))}{(1 - \mathrm{tr}(\tilde{\phi}\phi))^2} \\
&= \sup_{x\neq\pm 1} \frac{1 - (\frac{q}{q-1})^2(-\frac{q-2}{q}x^4 + \frac{2(q-2)}{q}x^2 + \frac{1}{q^2})}{(1 - \frac{q}{q-1}(x^2 - \frac{1}{q}))^2} \\
&= \sup_{x\neq\pm 1} \frac{\frac{q(q-2)}{(q-1)^2}(1-x^2)^2}{(\frac{q}{q-1})^2(1-x^2)^2} = \frac{q-2}{q}.
\end{aligned}$$

Lemma 3.4 *The critical radius θ_c of M_1 is*

$$\theta_c = \begin{cases} \tan^{-1}\sqrt{\dfrac{q}{q-2}} & \textit{for } q \geq 3, \\ \pi/2 & \textit{for } q = 2. \end{cases}$$

3.2 The proof of Theorem 2.2

3.2.1 Geometry of the manifold M_2

The index set M_2 is written as

$$M_2 = \Bigl\{\frac{1}{\sqrt{2}}HEH' \mid H \in V_{2,q}\Bigr\}, \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where

$$V_{2,q} = \{H : q \times 2 \mid H'H = I_2\}$$

is a Stiefel manifold. The dimension of the index set is

$$d = \dim(M_2) = \dim(V_{2,q}) = 2q - 3.$$

Since $\mathrm{tr}(HEH') = 0$, M_2 is also a subset of the linear subspace (9). It is easily shown that (9) is the linear hull of M_2 of dimension $p' = q(q+1)/2 - 1$.

In the following we use d and $2q-3$ interchangeably. We introduce a local coordinate system $t = (t^1, \ldots, t^d)$ for the sake of convenience of calculation. Each element of $H \in V_{2,q}$, $\phi \in M_2$ can be written as $H = H(t)$, $\phi = \phi(t)$. As in Section 3.1, derivative with respect to t^i is denoted by the subscript i, e.g., $H_i = \partial H/\partial t^i$, $\phi_{ij} = \partial^2\phi/\partial t^i \partial t^j$.

The tangent space $T_\phi(M_2)$ at $\phi = \phi(t)$ is spanned by

$$\phi_i = \frac{1}{\sqrt{2}}(H_i E H' + H E H_i'), \quad i = 1, \ldots, d.$$

The metric tensor of M_2 is given by

$$g_{ij} = \mathrm{tr}(\phi_i \phi_j) = \mathrm{tr}(EH'H_iEH'H_j) + \mathrm{tr}(H_i'H_j), \quad i,j = 1, \ldots, d. \tag{20}$$

Let $\bar{H}$ be a $q \times (q-2)$ matrix such that $(H, \bar{H})$ is orthogonal. Define a 2×2 matrix B_i and a $(q-2) \times 2$ matrix $C_i = (c_{i1}, c_{i2})$ by

$$H_i = (\, H \quad \bar{H} \,) \begin{pmatrix} B_i \\ C_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B_i \\ C_i \end{pmatrix} = \begin{pmatrix} H' \\ \bar{H}' \end{pmatrix} H_i. \tag{21}$$

Since B_i is skew symmetric we put $B_i = b_i J$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The metric (20) is rewritten as

$$\begin{aligned} g_{ij} &= \mathrm{tr}(EB_iEB_j) + \mathrm{tr}(B_i'B_j) + \mathrm{tr}(C_i'C_j) \\ &= 4b_ib_j + c_{i1}'c_{j1} + c_{i2}'c_{j2}, \qquad i,j = 1, \ldots, d. \end{aligned} \tag{22}$$

On the other hand, regarding $V_{2,q}$ as a submanifold of $R^{q\times 2}$ (the set of $q \times 2$ real matrices) endowed with the inner product $\mathrm{tr}(X'Y)$, $X, Y \in R^{q\times 2}$, we obtain the (pull-back) metric of $V_{2,q}$ as

$$\bar{g}_{ij} = \mathrm{tr}(H_i'H_j) = 2b_ib_j + c_{i1}'c_{j1} + c_{i2}'c_{j2}. \tag{23}$$

Let $d\phi$ and dH be denote the volume elements of M_2 and $V_{2,q}$, respectively. By comparing (22) and (23), we see that $\det(g_{ij}) = 2\det(\bar{g}_{ij})$ and hence $d\phi = \sqrt{2}\, dH$. Noting that the multiplicity of the map $H \mapsto \phi = HEH'/\sqrt{2}$ is 4, we have the following lemma.

Lemma 3.5 *The total volume of M_2 is given by*

$$\mathrm{Vol}(M_2) = \sqrt{2}\ \mathrm{Vol}(V_{2,q}) \times \frac{1}{4} = \frac{2^{q-1}\pi^{q-1}}{\Gamma(q-1)}.$$

Proof. The volume element of $V_{2,q}$ defined by the pull-back metric is $dH = \sqrt{2} \wedge_{i=1}^{2} \wedge_{j=i+1}^{q} h_j' dh_i$, where $H = (h_1, h_2)$ and $\bar{H} = (h_3, \dots, h_q)$ (Takemura and Kuriki (1996)). The total volume of $V_{2,q}$ is evaluated as

$$\mathrm{Vol}(V_{2,q}) = \sqrt{2} \int_{V_{2,q}} \bigwedge_{i=1}^{2} \bigwedge_{j=i+1}^{q} h_j' dh_i = \frac{2^{5/2} \pi^{q-1/2}}{\Gamma(\frac{q}{2})\,\Gamma(\frac{q-1}{2})} = \frac{2^{q+1/2} \pi^{q-1}}{\Gamma(q-1)}$$

(e.g., Muirhead (1982)). The proof is completed. ∎

It is easy to see that the orthogonal complement $(\mathrm{span}\{\phi\} \oplus T_\phi(M_2))^\perp$ in $\mathrm{Sym}(q)$ is a linear space of dimension $(q-1)(q-2)/2+1$ spanned by

$$\nu = \frac{a}{\sqrt{2}} HH' + \bar{H} A \bar{H}', \qquad a \in R, \quad A = (a_{ij}) \in \mathrm{Sym}(q-2). \tag{24}$$

The second derivative of ϕ is

$$\phi_{ij} = \frac{1}{\sqrt{2}} (H_{ij} E H' + H E H_{ij}' + H_i E H_j' + H_j E H_i').$$

Since $H' H_i + H_i' H = 0$ and $H' H_{ij} + H_{ij}' H + H_i' H_j + H_j' H_i = 0$, the inner product of ϕ_{ij} and ν in (24) is

$$\begin{aligned}
\mathrm{tr}(\nu \phi_{ij}) &= a \{-\mathrm{tr}(H_i E H_j') + \mathrm{tr}(H' H_i E H_j' H)\} + \sqrt{2}\, \mathrm{tr}(\bar{H}' H_i E H_j' \bar{H} A) \\
&= -a\, \mathrm{tr}(C_i E C_j') + \sqrt{2}\, \mathrm{tr}(C_i E C_j' A) \\
&= (\sqrt{2} b_i, c_{i1}', c_{i2}') \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a I_{q-2} + \sqrt{2} A & 0 \\ 0 & 0 & a I_{q-2} - \sqrt{2} A \end{pmatrix} \begin{pmatrix} \sqrt{2} b_j \\ c_{j1} \\ c_{j2} \end{pmatrix}.
\end{aligned}$$

On the other hand, since the metric g_{ij} is

$$g_{ij} = (\sqrt{2} b_i, c_{i1}', c_{i2}') \begin{pmatrix} \sqrt{2} b_j \\ c_{j1} \\ c_{j2} \end{pmatrix},$$

we have the following.

Lemma 3.6 *In an appropriate coordinate system, the second fundamental form of M_2 at ϕ with respect to the direction ν in (24) is written as*

$$H(\phi, \nu) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a I_{q-2} - \sqrt{2} A & 0 \\ 0 & 0 & -a I_{q-2} + \sqrt{2} A \end{pmatrix}.$$

3.2.2 The coefficients in the tube formula

The squared norm of ν in (24) is $\mathrm{tr}(\nu^2) = a^2 + \mathrm{tr}(A^2)$. This implies that if $A \in \mathrm{Sym}(q-1)$ is distributed as the multivariate symmetric normal distribution, and a is distributed as $N(0,1)$ independently of A, then ν in (24) is distributed as the multivariate standard normal distribution in the space $(\mathrm{span}\{\phi\} \oplus T_\phi(M_2))^\perp$. The proof of the following lemma is given in Appendix A.3.

Lemma 3.7

$$E[\mathrm{tr}_e H(\phi,\nu)] = \begin{cases} (-1)^{e/2} \dfrac{(q-2)!\, q!}{(q-2-e/2)!\,(q-e/2)!\,(e/2)!} & \textit{for } e \textit{ even,} \\ 0 & \textit{for } e \textit{ odd.} \end{cases}$$

As in the case of M_1, $E[\mathrm{tr}_e H(\phi,\nu)]$ is independent of ϕ. The integration in (30) with respect to $d\phi$ over M_2 reduces to multiplication by the constant $\mathrm{Vol}(M_2)$. Then by (30) the coefficient of the tube formula (29) for M_2 is given by

$$w_{2q-2-e} = \frac{\Gamma(q-1-e/2)}{2^{e/2+1}\,\pi^{q-1}} \mathrm{Vol}(M_2) \cdot E[\mathrm{tr}_e H],$$

which reduces to (6) in Theorem 2.2.

3.2.3 Critical radius of the manifold M_2

We obtain the critical radius θ_c of the manifold M_2 by virtue of Lemma A.1.

Let $\phi = (1/\sqrt{2})HEH'$ and $\tilde\phi = (1/\sqrt{2})\tilde H E \tilde H'$ be different points of M_2. The orthogonal projection of $\tilde\phi \in M_2$ onto $T_\phi(M_2)$ is given by

$$P_\phi(\tilde\phi) = \phi\, \mathrm{tr}(\phi\tilde\phi) + \sum_{i,j=1}^{d} \phi_i g^{ij} \mathrm{tr}(\phi_j \tilde\phi), \tag{25}$$

where g^{ij} is the (i,j)-th element of the inverse of the metric (g_{ij}) in (22). In the right hand side of (25),

$$\mathrm{tr}(\phi\tilde\phi) = \frac{1}{2}\mathrm{tr}(\tilde H E \tilde H' H E H') = \frac{1}{2}\mathrm{tr}(RER'E),$$

where $R = \tilde H' H$ is a 2×2 matrix. As in (21) define $B_i = b_i J$ and $C_i = (c_{i1}, c_{i2})$ so that $H_i = b_i H J + \bar H C_i$. Then

$$\mathrm{tr}(\phi_i\tilde\phi) = \mathrm{tr}(EH'\tilde H E \tilde H' H_i) = (2b_i, c_{i1}', c_{i2}') \begin{pmatrix} k \\ l_1 \\ l_2 \end{pmatrix},$$

where

$$k = \frac{1}{2}\mathrm{tr}(EH'\tilde H E \tilde H' H J) = \frac{1}{2}\mathrm{tr}(ER'ERJ),$$

and

$$L = (l_1, l_2) = \bar{H}'\tilde{H}E\tilde{H}'HE = \bar{H}'\tilde{H}ERE$$

is a $(q-2) \times 2$ matrix. Since

$$g_{ij} = (2b_i, c'_{i1}, c'_{i2}) \begin{pmatrix} 2b_j \\ c_{j1} \\ c_{j2} \end{pmatrix},$$

we have

$$\begin{aligned} \sum_{i,j=1}^{d} \mathrm{tr}(\phi_i \tilde{\phi}) g^{ij} \mathrm{tr}(\phi_j \tilde{\phi}) &= k^2 + l'_1 l_1 + l'_2 l_2 = k^2 + \mathrm{tr}(L'L) \\ &= \frac{1}{4}\mathrm{tr}(ER'ERJ)^2 + \mathrm{tr}(RR') - \mathrm{tr}(ERR'ERR'). \end{aligned}$$

Summarizing the above we have

$$\cot^2\theta_c = \sup_R \frac{1 - \frac{1}{4}\mathrm{tr}(RER'E)^2 - \frac{1}{4}\mathrm{tr}(ER'ERJ)^2 - \mathrm{tr}(RR') + \mathrm{tr}(ERR'ERR')}{(1 - \frac{1}{2}\mathrm{tr}(RER'E))^2}, \qquad (26)$$

where the supremum is taken over the set of 2×2 submatrices of any $q \times q$ orthogonal matrix such that

$$R = \tilde{H}'H \neq \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

In the case of $q = 2$,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \sin\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\sin\theta \end{pmatrix}, \quad 0 < \theta < \pi.$$

Then $\mathrm{tr}(RER'E) = 2\cos(2\theta)$, $\mathrm{tr}(ER'ERJ) = \pm 2\sin(2\theta)$, and $\cot^2\theta_c = \sup_\theta 0 = 0$.

In the case $q \geq 3$ put $R = (r_{ij})_{i,j=1,2}$. The argument of the supremum in (26) is written as

$$1 + \frac{\frac{1}{2}(\delta_1 - \delta_2)^2 + \delta_3}{\frac{1}{4}(\delta_1 + \delta_2)^2},$$

where

$$\delta_1 = 1 - r_{11}^2 + r_{21}^2, \quad \delta_2 = 1 - r_{22}^2 + r_{12}^2, \quad \delta_3 = -2(r_{12}^2 + r_{21}^2) + (r_{11}r_{12} - r_{21}r_{22})^2.$$

Noting that $|r_{11}| \leq 1$, $|r_{22}| \leq 1$, we have $|r_{11}r_{12} - r_{21}r_{22}| \leq \max(|r_{12} + r_{21}|, |r_{12} - r_{21}|)$, and hence $\delta_3 \leq -2(r_{12}^2 + r_{21}^2) + (r_{12} \pm r_{21})^2 = -(r_{12} \mp r_{21})^2 \leq 0$. Also noting that $\delta_1, \delta_2 \geq 0$, $\delta_1 + \delta_2 > 0$, we have

$$\cot^2\theta_c \leq 1 + 2\sup\left(\frac{\delta_1 - \delta_2}{\delta_1 + \delta_2}\right)^2 \leq 3.$$

Conversely, consider $R_0 = \mathrm{diag}(1, \cos\theta_0)$, $0 < \theta_0 < \pi$, as a 2×2 submatrix of a $q \times q$ orthogonal matrix. Then $\delta_1 = 0$, $\delta_2 = \sin^2\theta_0$, $\delta_3 = 0$, and hence $\cot^2\theta_c \geq 3$. Therefore $\cot^2\theta_c = 3$ for $q \geq 3$.

Lemma 3.8 *The critical radius θ_c of M_2 is*

$$\theta_c = \begin{cases} \pi/6 & \text{for } q \geq 3, \\ \pi/2 & \text{for } q = 2. \end{cases}$$

Appendix

A.1 The tube method

We give here a brief summary of the tube method from Section 3 of Kuriki and Takemura (1998).

Let M be a d-dimensional closed C^2-submanifold in the unit sphere S^{p-1} of R^p. Let $Z(u)$, $u = (u_1, \ldots, u_p)' \in M$, be a random field with the index set M defined by

$$Z(u) = u'z = \sum_{i=1}^{p} u_i z_i,$$

where $z = (z_1, \ldots, z_p)'$ is distributed according to the p-dimensional standard multivariate normal distribution $N_p(0, I_p)$. This is the canonical form of the Gaussian random field with a finite Karhunen-Loève expansion and a constant variance. The tube method is used for the purpose of obtaining the asymptotic expansion of the upper tail probability of the maximum

$$P(T \geq x), \qquad T = \max_{u \in M} Z(u), \tag{27}$$

as x goes to infinity.

The essential notions are the *tube* around M and the *critical radius* θ_c of M. The distance between two points $u, v \in S^{p-1}$ is given by $\arccos(u'v)$, which is the length of the part of the great circle joining u and v. For $0 < \theta < \pi$ the tube of geodesic distance θ around M on S^{p-1} is defined by

$$M_\theta = \Big\{ v \in S^{p-1} \mid \max_{u \in M} u'v > \cos\theta \Big\}.$$

For each $u \in M$ let $T_u(M) \in R^p$ denote the tangent space of M at u. Define a subset $C_\theta(u)$ of M_θ by

$$C_\theta(u) = \{ v \in M_\theta \mid u'v > \cos\theta \} \cap \{ u + T_u(M)^\perp \},$$

where $T_u(M)^\perp$ denotes the orthogonal complement of $T_u(M)$ in R^p. Since M is closed it holds obviously that

$$M_\theta = \bigcup_{u \in M} C_\theta(u). \tag{28}$$

It is said that M_θ does not have self-overlap if (28) gives a partition of M_θ. The critical radius θ_c of M is defined to be the supremum of θ such that M_θ does not have self-overlap.

By the compactness and the smoothness of M, we can prove that the critical radius θ_c is positive. Moreover, it can be evaluated by the the following lemma, which is the extension of Proposition 4.3 of Johansen and Johnstone (1990) to multidimensional cases.

Lemma A.1 *The critical radius θ_c of M is given by*

$$\cot^2\theta_c = \sup_{u,v \in M} \frac{1 - u'P_v u}{(1 - u'v)^2},$$

where P_v is the orthogonal projection onto the space $\mathrm{span}\{v\} \oplus T_v(M)$.

Let $H(u,v)$ denote the second fundamental form of M at u with respect to the direction $v \in (\mathrm{span}\{u\} \oplus T_u(M))^\perp$. Let $\mathrm{tr}_j H$ denote the j-th trace, i.e., the j-th elementary symmetric function of the eigenvalues of $H = H(u,v)$. Define $\mathrm{tr}_0 H = 1$.

The volume of M_θ, $\theta \le \theta_c$, is obtained by the tube formula below. In the following $\bar{B}_{m,n}(\cdot)$ denotes the upper tail probability of the beta distribution with parameter (m,n).

Lemma A.2 *For $0 \le \theta \le \theta_c$,*

$$\mathrm{Vol}(M_\theta) = \Omega_p \sum_{e=0,\, e:\mathrm{even}}^{d} w_{d+1-e} \bar{B}_{\frac{1}{2}(d+1-e),\frac{1}{2}(p-d-1+e)}(\cos^2\theta),$$

where

$$w_{d+1-e} = \frac{1}{\Omega_{d+1-e}\Omega_{p-d-1+e}} \int_M \Big[\int_{(\mathrm{span}\{u\}\oplus T_u(M))^\perp \cap S^{p-1}} \mathrm{tr}_e H(u,v)\, dv \Big]\, du. \tag{29}$$

Using the coefficients w_{d+1-e} in (29), the formula for the tail probability in (27) is given as follows.

Theorem A.1

$$P(T \ge x) = \sum_{e=0,\, e:\mathrm{even}}^{d} w_{d+1-e} \bar{G}_{d+1-e}(x^2) + O(\bar{G}_{p'}((1+\tan^2\theta_c)x^2)), \quad x \to \infty,$$

$p' = \dim(\mathrm{lin}\, M)$, where $\mathrm{lin}\, M$ is the linear hull of M in R^p.

Remark A.1 *The integral in (29) with respect to dv can be evaluated by introducing a random variable and taking its expectation. Let $V \in R^p$ be distributed as $N_p(0, I_p - P_u)$, where P_u is the $p \times p$ orthogonal projection matrix onto the $(d+1)$-dimensional linear subspace* $\mathrm{span}\{u\} \oplus T_u(M)$*. Then (29) is written as*

$$w_{d+1-e} = \frac{\Gamma(\frac{d+1-e}{2})}{2^{e/2+1}\pi^{(d+1)/2}} \int_M E[\mathrm{tr}_e H(u,V)]\, du. \tag{30}$$

A.2 Some moments in the multivariate symmetric normal distribution

We provide some lemmas concerning the moments of the multivariate symmetric normal distribution which are required in Appendix A.3 (the proof of Lemma 3.7).

Let $A = (a_{ij}) \in \mathrm{Sym}(p)$ be distributed according to the multivariate symmetric normal distribution. Let U, V and W be mutually disjoint subsets of the index set $\{1,2,\ldots,p\}$ of A. Put $u = |U|$, $v = |V|$ and $w = |W|$, the cardinalities of the sets. Let $A[U]$ denote the symmetric submatrix consisting of the elements a_{ij}, $i,j \in U$.

Define

$$Q(u,v,w) = E[\det A[U \cup W] \times \det A[V \cup W]], \tag{31}$$

and for y even define

$$R(y,w) = \frac{(-2)^{y/2+w}}{w!} \sum_{u+v=y} \binom{y}{u,v} Q(u,v,w). \tag{32}$$

We first give recurrence formulas for $Q(u,v,w)$ by combinatorial considerations.

Lemma A.3 *Let $(x)_i = x(x-1)\cdots(x-i+1)$. Define $(x)_0 = 1$ for all $x \geq 0$.*

$$\begin{aligned}
Q(0,0,0) &= Q(0,0,1) = 1, \\
Q(0,0,w) &= 2 \sum_{t=1,\, t:\text{odd}}^{w} \frac{(w-1)_{t-1}}{2^t} Q(0,0,w-t) \\
&\quad +3 \sum_{t=2,\, t:\text{even}}^{w} \frac{(w-1)_{t-1}}{2^t} Q(0,0,w-t), && (33)\\
Q(u,v,w) &= -\sum_{t=0,\, t:\text{even}}^{w} \frac{(u-1)\,(w)_t}{2^{t+1}} Q(u-2,v,w-t) \\
&\quad + \sum_{t=1,\, t:\text{odd}}^{w} \frac{v\,(w)_t}{2^{t+1}} Q(u-1,v-1,w-t) && (34)\\
&= -\sum_{t=0,\, t:\text{even}}^{w} \frac{(v-1)\,(w)_t}{2^{t+1}} Q(u,v-2,w-t) \\
&\quad + \sum_{t=1,\, t:\text{odd}}^{w} \frac{u\,(w)_t}{2^{t+1}} Q(u-1,v-1,w-t). && (35)
\end{aligned}$$

Proof. By completely expanding the determinants

$$\det A[U \cup W] \times \det A[V \cup W],$$

we have $(u+w)! \times (v+w)!$ terms. Each term has a zero or nonzero expectation. We consider here the characterization of terms with nonzero expectation. For notational convenience let B be the same matrix as A (i.e., $A = B$ a.s.), and consider the expansion of $\det A[U \cup W] \times \det B[V \cup W]$. For any particular term in the expansion, we consider a graph consisting of $u+v+w$ vertices and $(u+w)^2 + (v+w)^2$ directed edges. We identify the indices of U, V and W with the vertices. Therefore there are three kinds of vertices corresponding to U, V and W. Also we consider two kinds of directed edges. If the variable a_{ij} appears in the particular term, i and j are connected with a directed edge in solid line "$\longrightarrow$". (We call i the initial vertex, and j the terminal vertex. i and j may be identical.) Similarly if the variable b_{ij} appears in the term, i and j are connected by a directed edge in dashed line "$- \rightarrow$". Note that

- Each vertex of W is an initial vertex of both of a directed edge in solid line and a directed edge in dashed line, and is a terminal vertex of both of a directed edge in solid line and a directed edge in dashed line simultaneously.

- Each vertex of U is an initial vertex of a directed edge in solid line, and is a terminal vertex of a directed edge in solid line simultaneously.

- Each vertex of V is an initial vertex of a directed edge in dashed line, and is a terminal vertex of a directed edge in dashed line simultaneously.

Since the elements of A and B are zero-mean Gaussian random variables, the expectation of a particular term is nonzero if and only if any pair of the indices (i, j) (i and j may be identical) are connected by even numbers (may be 0) of edges. From now on consider the case where the term has a nonzero expectation. In this case if the pair (i, j) are connected, then one of the following holds.

- i and j are connected by a solid line and a dashed line ($i = j$, $i \neq j$).

- i and j are connected by two solid lines ($i \neq j$).

- i and j are connected by two dashed lines ($i \neq j$).

- i and j are connected by two solid lines and two dashed lines ($i \neq j$).

Each vertex of W has to be an initial or terminal vertex of four edges. On the other hand, two edges are needed to connect the vertex to another vertex. Therefore, each vertex of W has at most two adjacent vertices. Each vertex of U or V has to be an initial or terminal vertex of two edges. But any vertices of U or V without adjacent vertex do not appear in the terms with nonzero expectation. Therefore, each vertex of U or V has just one adjacent vertex.

From the considerations above, we see that the graph associated with the term with nonzero expectation consists of connected components (subgraphs) of the following eight types.

1. A component consisting of a single vertex of W. The vertex is connected with itself by a solid line and a dashed line.

2. A pair of two vertices of W. The two vertices are connected by two solid lines and two dashed lines.

3. A loop consisting of t (≥ 3) vertices of W. Two adjacent vertices are connected with a solid line and a dashed line. The directions of the two edges are the same.

4. A loop consisting of t (≥ 3) vertices of W. Two adjacent vertices are connected by a solid line and a dashed line. The directions of the two edges are reverse.

5. A loop consisting of t (≥ 4, even) vertices of W. Two adjacent vertices are connected by two solid lines or two dashed lines.

6. A chain consisting of two vertices of U as end points, and t (≥ 0, even) numbers of vertices of W as intermediate points. Two adjacent vertices are connected by two solid lines or two dashed lines.

7. A chain consisting of two vertices of V as end points, and t (≥ 0, even) numbers of vertices of W as intermediate points. Two adjacent vertices are connected by two solid lines or two dashed lines.

8. A chain consisting of a vertex of U and a vertex of V as end points, and t (≥ 1, odd) numbers of vertices of W as intermediate points.

Now we proceed to the proof of (33). Fix an index i_0 of W. We evaluate the contribution of the case where the vertex i_0 is contained in a particular type of the connected subgraphs to $Q(0,0,w) = E[\det A[W] \times \det B[W]]$. The connected subgraph containing the vertex i_0 has to be of the types 1–5. In the following the sign of a cycle is denoted by $\mathrm{sgn}(\cdot)$.

- The case where i_0 itself forms a connected graph (type 1). The contribution to $Q(0,0,w)$ is

$$E[a_{i_0 i_0} b_{i_0 i_0}] Q(0,0,w-1) = Q(0,0,w-1).$$

- The case where the pair of i_0 and the other index $i_1 \in W \setminus \{i_0\}$ form a connected graph (type 2). The contribution to $Q(0,0,w)$ is

$$\mathrm{sgn}(i_0\ i_1)^2 \sum_{i_1 \neq i_0} E[a_{i_0 i_1} a_{i_1 i_0} b_{i_0 i_1} b_{i_1 i_0}] Q(0,0,w-2) = \frac{3(w-1)}{2^2} Q(0,0,w-2).$$

- The case where $i_0, i_1, \ldots, i_{t-1}$ ($t \geq 3$) form a type 3 loop. There are $(w-1)_{t-1}$ ways to make a loop. Each loop has an expectation

$$\mathrm{sgn}(i_0\ i_1\ \cdots\ i_{t-1})^2 E[a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{t-1} i_0} b_{i_0 i_1} b_{i_1 i_2} \cdots b_{i_{t-1} i_0}] = 1/2^t.$$

The contribution to $Q(0,0,w)$ is

$$\frac{(w-1)_{t-1}}{2^t} Q(0,0,w-t) \qquad (t \geq 3).$$

- The case where $i_0, i_1, \ldots, i_{t-1}$ ($t \geq 3$) form a type 4 loop. There are $(w-1)_{t-1}$ ways to make a loop. Each loop has an expectation

$$\mathrm{sgn}(i_0\ i_1\ \cdots\ i_{t-1})\, \mathrm{sgn}(i_0\ i_{t-1}\ \cdots\ i_1) \times E[a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{t-1} i_0} b_{i_0 i_{t-1}} b_{i_{t-1} i_{t-2}} \cdots b_{i_1 i_0}] = 1/2^t.$$

The contribution to $Q(0,0,w)$ is

$$\frac{(w-1)_{t-1}}{2^t} Q(0,0,w-t) \qquad (t \geq 3).$$

- The case where $i_0, i_1, \ldots, i_{t-1}$ ($t \geq 4$, even) form a type 5 loop. There are $(w-1)_{t-1}$ ways to make a loop. Each loop has an expectation

$$\begin{aligned}&\mathrm{sgn}(i_0\ i_1)\,\mathrm{sgn}(i_1\ i_2)\cdots\mathrm{sgn}(i_{t-1}\ i_0)\\ &\quad\times E[a_{i_0 i_1}a_{i_1 i_0}b_{i_1 i_2}b_{i_2 i_1}\cdots a_{i_{t-2}i_{t-1}}a_{i_{t-1}i_{t-2}}b_{i_{t-1}i_0}b_{i_0 i_{t-1}}] = (-1/2)^t.\end{aligned}$$

  The contribution to $Q(0,0,w)$ is

$$\frac{(w-1)_{t-1}}{2^t}Q(0,0,w-t) \qquad (t \geq 4,\ \text{even}).$$

Summing up the above five cases, we get (33).

Next we proceed to the proof of (34). Fix an element i_0 of U. We evaluate the contribution of the case where the vertex i_0 is contained in a particular type of the connected subgraphs to $Q(u,v,w) = E[\det A[U \cup W] \times \det B[V \cup W]]$. The connected subgraph containing the index i_0 has to be of the types 6, 8.

- The case where $i_0, i_1, \ldots, i_t, i_{t+1}$ ($t \geq 0$, even) form a type 6 chain. There are $w_t \times (u-1)$ ways to make a chain. Each chain has an expectation

$$\mathrm{sgn}(i_0\ i_1)\cdots\mathrm{sgn}(i_t\ i_{t+1})E[a_{i_0 i_1}a_{i_1 i_0}b_{i_1 i_2}b_{i_2 i_1}\cdots a_{i_t i_{t+1}}a_{i_{t+1}i_t}] = (-1/2)^{t+1}.$$

  The contribution to $Q(u,v,w)$ is

$$-\frac{(u-1)\,w_t}{2^{t+1}}Q(u-2,v,w) \qquad (t \geq 0,\ \text{even}).$$

- The case where $i_0, i_1, \ldots, i_t, i_{t+1}$ ($t \geq 1$, odd) form a type 8 chain. There are $w_t \times v$ ways to make a chain. Each chain has an expectation

$$\mathrm{sgn}(i_0\ i_1)\cdots\mathrm{sgn}(i_t\ i_{t+1})E[a_{i_0 i_1}a_{i_1 i_0}b_{i_1 i_2}b_{i_2 i_1}\cdots b_{i_t i_{t+1}}b_{i_{t+1}i_t}] = (-1/2)^{t+1}.$$

  The contribution to $Q(0,0,w)$ is

$$\frac{v\,w_t}{2^{t+1}}Q(u-1,v-1,w) \qquad (t \geq 1,\ \text{odd}).$$

Summing up the two cases above, we get (34). The proof of (35) is parallel to that of (34) and omitted. ∎

As a corollary to Lemma A.3, we obtain recurrence formulas for $R(y,w)$ of (32).

Corollary A.1

$$\begin{aligned}R(0,0) &= 1, \quad R(0,1) = -2,\\ R(0,w) &= -\frac{2}{w}\sum_{t=1,\,t:\text{odd}}^{w} R(0,w-t) + \frac{3}{w}\sum_{t=2,\,t:\text{even}}^{w} R(0,w-t), \qquad (36)\\ R(y,w) &= 2(y-1)\sum_{t=0}^{w} R(y-2,w-t). \qquad (37)\end{aligned}$$

Proof. (36) follows from (33). (37) follows from

$$\begin{aligned}(u+v)Q(u,v,w) &= -\sum_{t=0,\,t:\text{even}}^{w} \frac{u(u-1)\,(w)_t}{2^{t+1}} Q(u-2,v,w-t)\\ &\quad +2\sum_{t=1,\,t:\text{odd}}^{w} \frac{uv\,(w)_t}{2^{t+1}} Q(u-1,v-1,w-t)\\ &\quad -\sum_{t=0,\,t:\text{even}}^{w} \frac{v(v-1)\,(w)_t}{2^{t+1}} Q(u,v-2,w-t).\end{aligned}$$

∎

For positive integer m write

$$m!! = \begin{cases} m(m-2)\cdots 1 & (m:\text{odd}),\\ m(m-2)\cdots 2 & (m:\text{even}).\end{cases}$$

We also define $0!! = 1$.

Lemma A.4 *Let A be distributed according to the $p\times p$ multivariate symmetric normal distribution. Then*

$$E[\det(A)^2] = Q(0,0,p) = \begin{cases} 2^{-p}\,\dfrac{2}{3}\,(p+2)!!\,p!! & (p:\mathit{odd}),\\ 2^{-p}\,\dfrac{2p+3}{3}\,(p+1)!!\,(p-1)!! & (p:\mathit{even}),\end{cases}$$

or equivalently

$$R(0,p) = \begin{cases} -\dfrac{2}{3}\dfrac{(p+2)!!}{(p-1)!!} & (p:\mathit{odd}),\\ \dfrac{2p+3}{3}\dfrac{(p+1)!!}{p!!} = \dfrac{1}{3}\dfrac{(p+1)!!}{(p-2)!!} + \dfrac{1}{3}\dfrac{(p+3)!!}{p!!} & (p:\mathit{even}).\end{cases} \qquad (38)$$

Proof. For nonnegative integer h and nonnegative even integer k, define

$$S_k^h = \frac{(k+h)!!}{k!!}.$$

Then it is easily shown that

$$S_k^{h+2} - S_{k-2}^{h+2} = (h+2)S_k^h, \qquad \sum_{t=0,t:\text{even}}^{k} S_t^h = \frac{1}{h+2} S_k^{h+2}.$$

In order to prove the lemma, we only have to show that

$$R(0,p) = \begin{cases} -\dfrac{2}{3} S_{p-1}^3 & (p:\text{odd}),\\ \dfrac{1}{3} S_{p-2}^3 + \dfrac{1}{3} S_p^3 & (p:\text{even}),\end{cases}$$

satisfies the recurrence formula (36).

When p is even,

$$
\begin{aligned}
-\frac{2}{p}\sum_{t=1,t:\text{odd}}^{p} R(0,p-t)+\frac{3}{p}\sum_{t=2,t:\text{even}}^{p} R(0,p-t) \\
&= \frac{4}{3p}\sum_{l=1,l:\text{odd}}^{p} S_{l-1}^3+\frac{1}{p}\sum_{l=0,l:\text{even}}^{p-2}(S_{l-2}^3+S_l^3) \\
&= \frac{4}{15p}S_{p-2}^5+\frac{1}{5p}(S_{p-4}^5+S_{p-2}^5) \\
&= \frac{1}{3}S_{p-2}^3+\frac{1}{3}S_p^3=R(0,p).
\end{aligned}
$$

When p is odd,

$$
\begin{aligned}
-\frac{2}{p}\sum_{t=1,t:\text{odd}}^{p} R(0,p-t)+\frac{3}{p}\sum_{t=2,t:\text{even}}^{p} R(0,p-t) \\
&= -\frac{2}{3p}\sum_{l=0,l:\text{even}}^{p}(S_{l-2}^3+S_l^3)-\frac{2}{p}\sum_{l=1,l:\text{odd}}^{p-2} S_{l-1}^3 \\
&= -\frac{2}{15p}(S_{p-3}^5+S_{p-1}^5)-\frac{2}{5p}S_{p-3}^5 \\
&= -\frac{2}{3}S_{p-1}^3=R(0,p).
\end{aligned}
$$

The proof is completed. ∎

A.3 Proof of Lemma 3.7

Let $A=A_p\in\mathrm{Sym}(p)$ be a multivariate symmetric normal random matrix, and let $a\in R$ be a standard normal random variable independent of A. Let

$$
H=\begin{pmatrix} aI_p-\sqrt{2}A_p & 0 \\ 0 & -aI_p+\sqrt{2}A_p \end{pmatrix}.
$$

Comparing the coefficients of x^{2p-e} in

$$
\begin{aligned}
\sum_{e=0}^{2p} x^{2p-e}\mathrm{tr}_e H &= \det(xI_{2p}+H)=\det(x^2I_p-(aI_p-\sqrt{2}A_p)^2) \\
&= \sum_{e=0}^{p} x^{2(p-e)}(-1)^e\mathrm{tr}_e(aI_p-\sqrt{2}A_p)^2,
\end{aligned}
$$

we have

$$
E[\mathrm{tr}_e H]=\begin{cases} (-1)^{e/2}E[\mathrm{tr}_{e/2}(aI_p-\sqrt{2}A_p)^2] & \text{for } e \text{ even}, \\ 0 & \text{for } e \text{ odd}. \end{cases}
$$

Let $D(p,e)$ denote the expectation of the $e\times e$ ($e\le p$) principal minor of the matrix $(aI_p-\sqrt{2}A_p)^2$ consisting of the first e rows and the first e columns. Then

$$E[\mathrm{tr}_e(aI_p-\sqrt{2}A_p)^2]=\binom{p}{e}D(p,e).$$

Therefore, in order to prove Lemma 3.7, we have to show that

$$D(p,e)=(p+2)_e=(p+2)!/(p+2-e)! \qquad (0\le e\le p). \tag{39}$$

Let B be a $e\times(p-e)$ random matrix consisting of $e\times(p-e)$ i.i.d. standard normal random variables. Then

$$\begin{aligned} D(p,e) &= E\Big[\det(aI_e-\sqrt{2}A_e,\ -B)\begin{pmatrix} aI_e-\sqrt{2}A_e \\ -B' \end{pmatrix}\Big] \\ &= E[\det((aI_e-\sqrt{2}A_e)^2+BB')]. \end{aligned} \tag{40}$$

Note that

$$BB'\sim W_e(p-e,I_e),$$

the $e\times e$ Wishart distribution with $p-e$ degrees of freedom.

The determinant of the sum of two matrices C, D is expressed as

$$\det(C+D)=\sum_{J,K}\pm\det C[J,K]\det D[\bar{J},\bar{K}],$$

where J, K are subsets of the index set, and $\bar{J}$, $\bar{K}$ are their complements. $C[J,K]$ is the submatrix of C consisting of the rows and columns of C labeled J and K, respectively. For the matrix B in (40), we can show that

$$E[\det((BB')[J,K])]=0 \quad (J\ne K).$$

This is because for a partition $B'=(B_1',B_2',B_3')$, we see

$$\begin{aligned} &E\Big[\det\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}(B_1',\ B_3')\Big]=E\Big[\det\begin{pmatrix} B_1B_1' & B_1B_3' \\ B_2B_1' & B_2B_3' \end{pmatrix}\Big] \\ &\quad = E[\det(B_1B_1')\,E[\det(B_2(I-B_1'(B_1B_1')^{-1}B_1)B_3')\mid B_1]]=0 \end{aligned}$$

by, e.g., the Binet-Cauchy formula.

Therefore, (40) can be rewritten as follows.

$$\begin{aligned} D(p,e) &= \sum_{f=0}^{e}\binom{e}{f}D(e,f)\,E[\det W_{e-f}(p-e,I_{e-f})] \\ &= \sum_{f=\max(0,2e-p)}^{e}\binom{e}{f}D(e,f)\,(p-e)_{e-f} \qquad (0\le f\le e\le p). \end{aligned} \tag{41}$$

Here we use $E[\det W_m(n,I_m)]=(n)_m$. Note that $(n)_m=0$ $(n<m)$.

On the other hand, comparing the coefficients of x^e in the identity

$$(1+x)^{p+2} = (1+x)^{e+2}(1+x)^{p-e},$$

and multiplying them by $e!$, we get

$$(p+2)_e = \sum_{f=\max(0,2e-p)}^{e} \binom{e}{f} (e+2)_f (p-e)_{e-f}. \tag{42}$$

If we can show that

$$D(p,p) = (p+2)_p = (p+2)!/2 \tag{43}$$

holds for any p, then (39) can be proved by mathematical induction on p by comparing (41) and (42).

Now it remains to prove (43). In order to prove

$$\frac{D(p,p)}{p!} = \binom{p+2}{2} = (-1)^p \binom{-3}{p},$$

we will show that

$$G_D(x) = \sum_{p=0}^{\infty} x^p \frac{D(p,p)}{p!} = (1-x)^{-3}.$$

Let Q, R be defined by (31), (32) in Appendix A.2. Then

$$\begin{aligned}
D(p,p) &= E[\det(aI_p - \sqrt{2}A_p)^2] \\
&= E\Big[\det\Big(aI_{2p} + \begin{pmatrix} -\sqrt{2}A_p & 0 \\ 0 & -\sqrt{2}A_p \end{pmatrix}\Big)\Big] \\
&= \sum_{U,V,W} E[a^{(p-u-w)+(p-v-w)}]\,(-\sqrt{2})^{(u+w)+(v+w)}\, Q(u,v,w) \\
&= \sum_{0\le u+v+w\le p\ (u+v:\text{even})} \binom{p}{u,v,w,p-u-v-w} (2p-2w-u-v-1)!! \\
&\qquad \times 2^{(u+v)/2+w}\, Q(u,v,w) \\
&= \sum_{0\le y+w\le p\ (y:\text{even})} \frac{p!\,(2p-2w-y-1)!!}{y!\,(p-y-w)!} (-1)^{y/2+w} R(y,w).
\end{aligned} \tag{44}$$

Multiply the right hand side of (44) by $x^p/p!$, and take a summation with respect to p. For y, w fixed, the coefficients of $(1/y!)\,(-1)^{y/2+w}\,R(y,w)$ in the summation is

$$\begin{aligned}
\sum_{p=y+w}^{\infty} \frac{(2p-2w-y-1)!!}{(p-y-w)!} x^p &= x^{y+w} \sum_{r=0}^{\infty} \frac{(2r+y-1)!!}{r!} x^r \qquad (r = p-y-w) \\
&= x^{y+w} \sum_{r=0}^{\infty} (y-1)!! \frac{\frac{2r+y-1}{2}\frac{2r-y-3}{2}\cdots\frac{y+1}{2}}{r!} (2x)^r \\
&= x^{y+w}(y-1)!!(1-2x)^{-(y+1)/2}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
G_D(x) &= \sum_{y,w\geq 0,\ y:\text{even}} x^{y+w}(y-1)!!(1-2x)^{-(y+1)/2}\frac{1}{y!}(-1)^{y/2+w}R(y,w) \\
&= \sum_{y\geq 0,\ y:\text{even}} x^{y}(1-2x)^{-(y+1)/2}\frac{(-1)^{y/2}}{y!!}G_R(-x;y), \qquad (45)
\end{aligned}
$$

where we put

$$
G_R(z;y) = \sum_{w=0}^{\infty} z^w R(y,w),
$$

a generating function of $R(y,w)$ with respect to w. By virtue of the recurrence relation (37),

$$
\begin{aligned}
G_R(z;y) &= 2(y-1)\sum_{w=0}^{\infty} z^w \sum_{t=0}^{w} R(y-2,w-t) \\
&= 2(y-1)\sum_{0\leq t\leq w} z^t z^{w-t} R(y-2,w-t) \\
&= 2(y-1)\sum_{t=0}^{\infty} z^t G_R(z;y-2) \\
&= 2(y-1)(1-z)^{-1}G_R(z;y-2).
\end{aligned}
$$

Using this iteratively, we get

$$
G_R(z;y) = 2^{y/2}(y-1)!!(1-z)^{-y/2}G_R(z;0).
$$

Also by (38),

$$
\begin{aligned}
G_R(z;0) &= \sum_{w=0}^{\infty} z^w R(0,w) \\
&= \sum_{w:\text{even}} z^w\Big(\frac{1}{3}\frac{(w+1)!!}{(w-2)!!}+\frac{1}{3}\frac{(w+3)!!}{w!!}\Big) - \sum_{w:\text{odd}} z^w \frac{2}{3}\frac{(w+2)!!}{(w-1)!!} \\
&= (z^2+1-2z)\frac{1}{3}\sum_{w:\text{even}} z^w \frac{(w+3)!!}{w!!} \\
&= (z-1)^2\frac{1}{3}\sum_{w:\text{even}} (z^2)^{w/2} 1\cdot 3\cdot \frac{\frac{5}{2}\frac{7}{2}\cdots\frac{w+3}{2}}{(w/2)!} \\
&= (z-1)^2(1-z^2)^{-5/2} = (1-z)^{-1/2}(1+z)^{-5/2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G_R(z;y) &= 2^{y/2}(y-1)!!(1-z)^{-y/2}(1-z)^{-1/2}(1+z)^{-5/2} \\
&= 2^{y/2}(y-1)!!(1-z)^{-(y+1)/2}(1+z)^{-5/2}.
\end{aligned}
$$

Substituting this into (45), we have

$$
\begin{aligned}
G_D(x) &= (1-x)^{-5/2} \sum_{y:\text{even}} \frac{(y-1)!!}{y!!}(-2)^{y/2} x^y \{(1-2x)(1+x)\}^{-(y+1)/2} \\
&= (1-x)^{-5/2}\{(1-2x)(1+x)\}^{-1/2} \sum_{y:\text{even}} \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{y-1}{2}}{(y/2)!}\Big(-\frac{2x^2}{(1-2x)(1+x)}\Big)^{y/2} \\
&= (1-x)^{-5/2}\{(1-2x)(1+x)\}^{-1/2}\Big(1+\frac{2x^2}{(1-2x)(1+x)}\Big)^{-1/2} \\
&= (1-x)^{-3}.
\end{aligned}
$$

The proof of Lemma 3.7 is completed.

References

[1] Anderson, T. W. and Stephens, M. A. (1972). Tests for randomness of directions against equatorial and bimodal alternatives. *Biometrika*, **59**, 613–621.

[2] Hotelling, H. (1939). Tubes and spheres in n-spaces, and a class of statistical problems. *Am. J. Math.*, **61**, 440–460.

[3] Johansen, S. and Johnstone, I. (1990). Hotelling's theorem on the volume of tubes: Some illustrations in simultaneous inference and data analysis. *Ann. Statist.*, **18**, 652–684.

[4] Knowles, M. and Siegmund, D. (1989). On Hotelling's approach to testing for a nonlinear parameter in regression. *Internat. Statist. Rev.*, **57**, 205–220.

[5] Kuriki, S. and Takemura, A. (1998). Tail probabilities of the maxima of multilinear forms and their applications. *Discussion Paper CIRJE-F-4*, Faculty of Economics, Univ. of Tokyo.

[6] Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory*. Wiley, New York.

[7] Naiman, D. Q. (1990). Volumes of tubular neighborhoods of spherical polyhedra and statistical inference. *Ann. Statist.*, **18**, 685–716.

[8] Park, M. G. and Sun, J. (1998). Tests in projection pursuit regression. *J. Statist. Plann. Inference*, **75**, 65–90.

[9] Siotani, M., Hayakawa, T., and Fujikoshi, Y. (1985). *Modern Multivariate Analysis: A Graduate Course and Handbook*, American Science Press, Columbus.

[10] Sun, J. (1991). Significance levels in exploratory projection pursuit. *Biometrika*, **78**, 759–769.

[11] Sun, J. (1993). Tail probabilities of the maxima of Gaussian random fields. *Ann. Probab.*, **21**, 34–71.

[12] Takemura, A. and Kuriki, S. (1996). Theory of cross sectionally contoured distributions and its applications. *Discussion Paper 96-F-15*, Faculty of Economics, Univ. of Tokyo.

[13] Takemura, A. and Kuriki, S. (1999). Maximum of Gaussian field on piecewise smooth domain: Equivalence of tube method and Euler characteristic method. *Discussion Paper CIRJE-F-54*, Faculty of Economics, Univ. of Tokyo.

[14] Watson, G. S. (1983). *Statistics on Spheres*. Wiley, New York.

[15] Weyl, H. (1939). On the volume of tubes. *Am. J. Math.*, **61**, 461–472.

Table 2.1. Asymptotic power comparisons.

(Monte Carlo simulations with 200,000 replications.)

Δ	S_{max} α=0.05	S_{min} α=0.05	S_{range} α=0.05	S_{max} α=0.01	S_{min} α=0.01	S_{range} α=0.01
Δ_0	0.050	0.050	0.050	0.010	0.010	0.010
$\frac{1}{4}\Delta_1$	0.053	0.053	0.053	0.011	0.011	0.011
$\frac{1}{2}\Delta_1$	0.061	0.060	0.062	0.013	0.013	0.013
Δ_1	0.101	0.083	0.098	0.028	0.020	0.026
$2\Delta_1$	0.316	0.190	0.288	0.140	0.060	0.118
$3\Delta_1$	0.668	0.379	0.612	0.438	0.157	0.368
$4\Delta_1$	0.916	0.626	0.880	0.787	0.338	0.710
$\frac{1}{4}\Delta_2$	0.052		0.053	0.011		0.011
$\frac{1}{2}\Delta_2$	0.060		0.061	0.013		0.013
Δ_2	0.092		0.098	0.024		0.026
$2\Delta_2$	0.255		0.289	0.100		0.120
$3\Delta_2$	0.550		0.624	0.312		0.385
$4\Delta_2$	0.826		0.890	0.629		0.734

$$\Delta_0 = \text{diag}(0,0,0), \quad \Delta_1 = \text{diag}(2,-1,-1)/\sqrt{6}$$
$$\Delta_2 = \text{diag}(1,0,-1)/\sqrt{2}$$

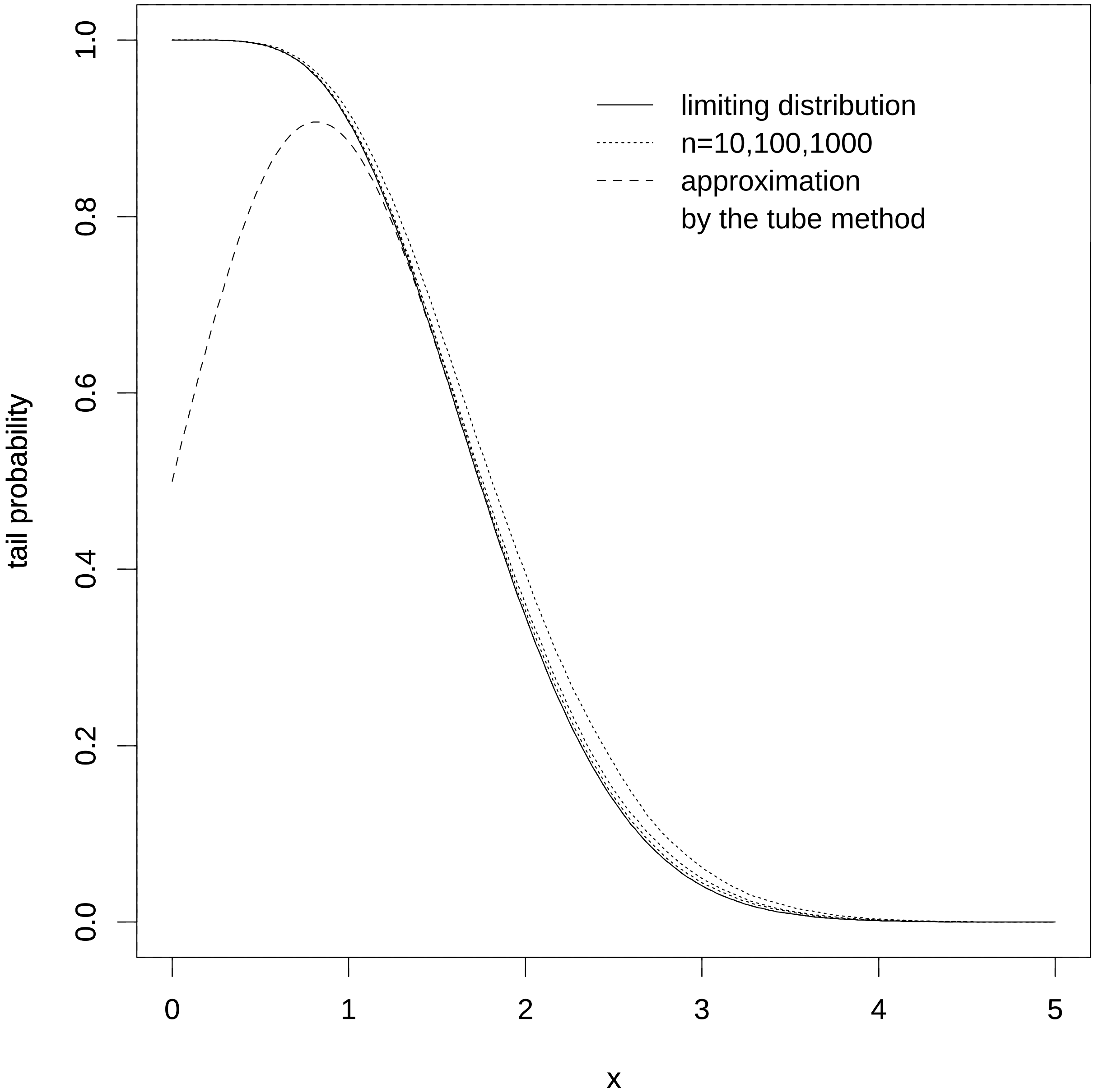

Figure 2.1. Tail probabilities of $S_{\max}$ when $q = 3$.
($n = 10, 100, 1000, \infty$ and approximation by the tube method.)

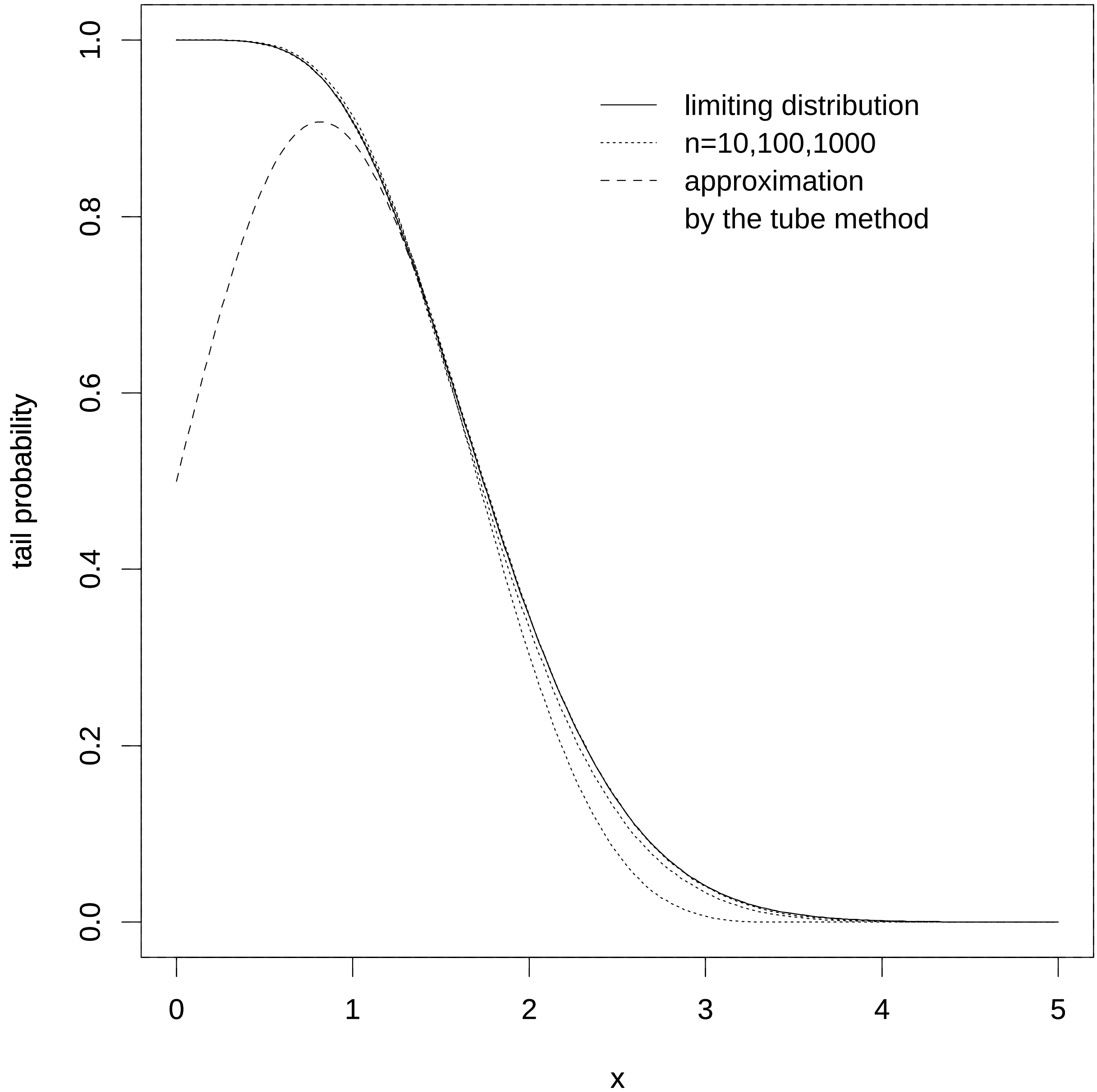

Figure 2.2. Tail probabilities of $S_{\min}$ when $q = 3$.
($n = 10, 100, 1000, \infty$ and approximation by the tube method.)

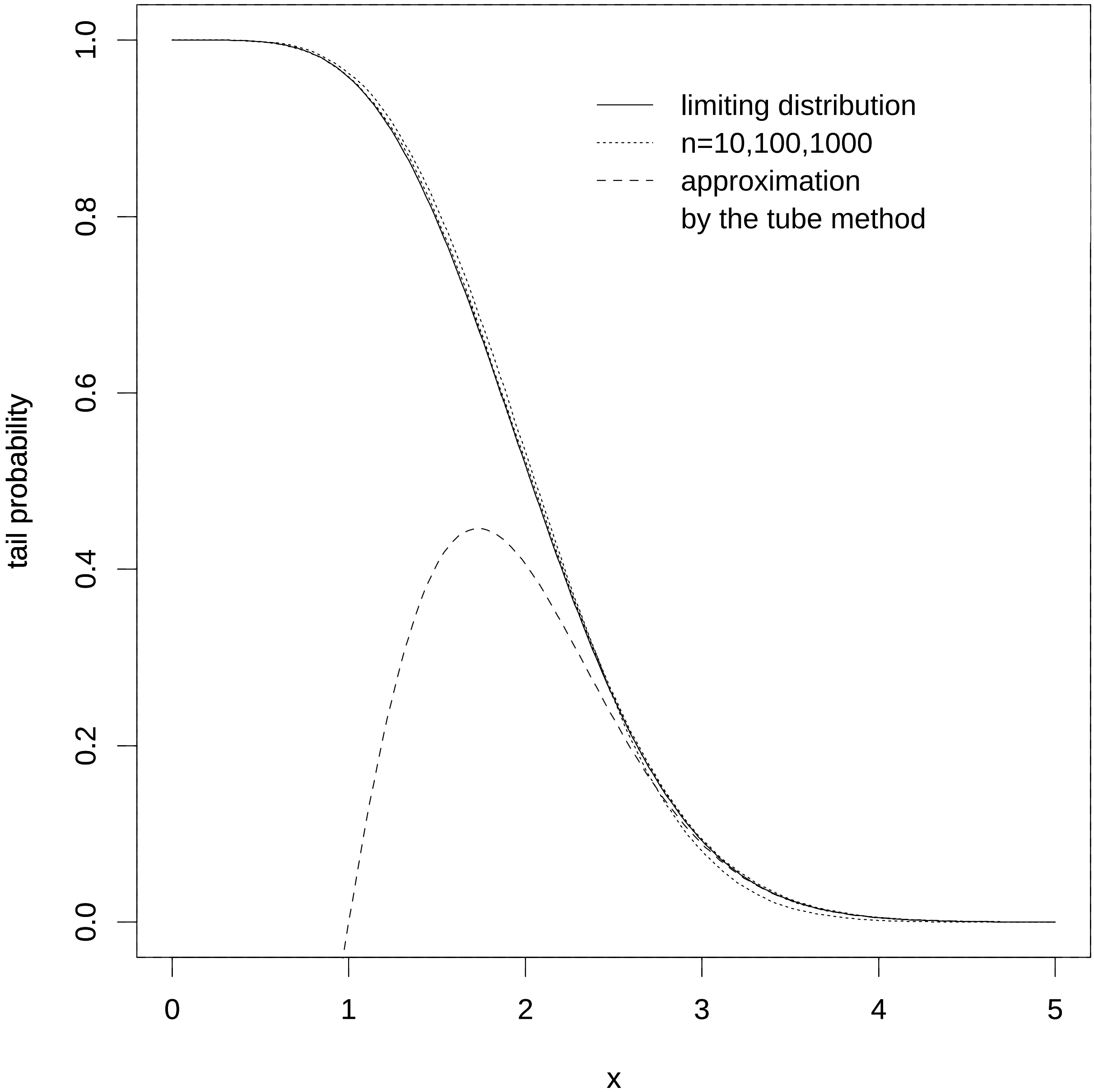

Figure 2.3. Tail probabilities of $S_{\rm range}$ when $q = 3$.
($n = 10, 100, 1000, \infty$ and approximation by the tube method.)