The Irrelevance of Market Incompleteness for the Price of Aggregate Risk*

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Abstract

In models with a large number of agents who have constant relative risk aversion (CRRA) preferences, the absence of insurance markets for idiosyncratic labor income risk has no effect on the premium for aggregate risk if the distribution of idiosyncratic risk is independent of aggregate shocks. In spite of the missing markets, a representative agent who consumes aggregate consumption prices the excess returns on stocks correctly. This result holds regardless of the persistence of the idiosyncratic shocks, as long as they are not permanent, even when households face binding, and potentially very tight borrowing constraints. Consequently, in this class of models there is no link between the extent of self-insurance against idiosyncratic income risk and aggregate risk premia.

1 Introduction

This paper examines whether closing down insurance markets for idiosyncratic risk increases the risk premium that stocks command over bonds, and we provide general conditions under which it does not.

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We study a standard incomplete markets model populated by a continuum of agents who have CRRA preferences and who can only trade a risk-free bond and a stock. The presence of uninsurable idiosyncratic risk is shown to lower the equilibrium risk-free rate, but it has no effect on the price of aggregate risk in equilibrium if the distribution of idiosyncratic shocks is statistically independent of aggregate shocks. Consequently, in this class of models, the representative agent Consumption-CAPM (CCAPM) developed by Breeden (1979) and Lucas (1978) prices the excess returns on the stock correctly. Therefore, as long as idiosyncratic shocks are distributed independently of aggregate shocks, the extent to which households manage to insure against idiosyncratic income risk is irrelevant for risk premia. These results deepen the equity premium puzzle, because we show that Mehra and Prescott's (1985) statement of the puzzle applies to a much larger class of incomplete market models.[1]

Our result holds regardless of the persistence of the idiosyncratic shocks, as long as these shocks are not completely permanent, and it is robust to the introduction of various forms of borrowing and solvency constraints, regardless of the tightness of these constraints. Our result also survives when agents have non-time-additive preferences if the consumption aggregator is a homogeneous function.

In addition, we show that adding markets does not always lead to more risk sharing. In particular, if the logarithm of aggregate consumption follows a random walk, allowing agents to trade claims on payoffs that are contingent on *aggregate shocks,* in addition to the risk-free bond and the stock, does not help them to smooth their consumption. Agents only trade the stock to smooth their consumption, and introducing these additional state contingent claims leaves interest rates and asset prices unaltered. However, if there is predictability in aggregate consumption growth, agents want to hedge their portfolio against interest rate shocks, creating a role for trade in a richer menu of assets. The risk premium irrelevance result, however, still applies. Finally, we also show that idiosyncratic uninsurable income risk does not contribute any variation in the conditional market price of risk, beyond what is built into aggregate consumption growth.

Most related to our paper is Constantinides and Duffie (henceforth CD) (1996), who consider an environment in which agents face only permanent idiosyncratic income shocks and can trade stocks and bonds. Their equilibrium is characterized by the absence of trade in financial markets. By choosing the right stochastic household income process, CD show they can deliver autarchic equilibrium asset prices with all desired properties. CD's results also imply that if the cross-sectional variance of consumption growth is orthogonal to returns, then the equilibrium risk premium is equal to the one in the representative agent model. We show that this characterization of the household consumption process is indeed the correct

[1] Weil's (1989) statement of the risk-free rate puzzle, on the contrary, does not.

one in equilibrium in a large class of incomplete market models.[2] Relative to CD, our paper adds potentially binding solvency or borrowing constraints and the equilibrium does feature trade in financial markets, but we do not make any assumptions about the distribution of the underlying shocks other than mean reversion. The consumption growth distribution across household is the endogenous, equilibrium outcome of these trades, but we can still fully characterize equilibrium asset prices.[3]

In the quest towards the resolution of the equity premium puzzle identified by Hansen and Singleton (1983) and Mehra and Prescott (1985), uninsurable idiosyncratic income risk has been introduced into standard dynamic general equilibrium models.[4] Incomplete insurance of household consumption against idiosyncratic income risk was believed to increase the aggregate risk premium. Heaton and Lucas (1996) attribute the failure of incomplete market models (or their partial success) in matching moments of asset prices to the household's success in smoothing consumption in this class of models. In contrast, our paper shows *analytically* that there is essentially no link between the extent to which idiosyncratic risk can be traded away by households and the size of the risk premium. The main contribution of our paper is to argue that Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Marcet and Singleton (1999) and others in this literature have reached the right conclusions -namely that adding uninsurable idiosyncratic income risk to standard models does not alter the asset pricing implications of the model - but not for the right reasons. As long as the distribution of idiosyncratic shocks is independent of aggregate shocks, the extent of self-insurance does not matter. We can make our solvency constraints arbitrarily tight or make the income process highly persistent, and our theoretical result still goes through. In equilibrium, all households bear the same amount of aggregate risk, and accordingly, they are only compensated in equilibrium for the aggregate consumption growth risk they take on by investing in stocks.

Most of the work on incomplete markets and risk premia documents the moments of model-generated data for particular calibrations, but there are few analytical results. Levine and Zame (2002) show that in economies populated by agents with infinite horizons, the

[2]Krebs (2005) derives the same result in a production economy with human capital accumulation. In his world, as in ours, the wealth distribution is not required as a state variable to characterize equilibria, in spite of the presence of aggregate shocks. As in CD (1996), however, the equilibrium is autarkic so that households cannot diversify any of their risk in equilibrium. Finally, Lettau (2006) also shows that *if* household consumption (in logs) consists of an aggregate and an idiosyncratic part, the latter does not affect risk premia.

[3]In a separate class of continuous-time diffusion models, Grossman and Shiller (1982) have demonstrated that heterogeneity has no effect on risk premia, simply because the cross-sectional variance of consumption growth is locally deterministic. Our irrelevance result is obtained in a different class of discrete-time incomplete market models in which heterogeneity does potentially matter, as shown by Mankiw (1986), CD and others, but we find the model itself cannot endogenously activate the Mankiw-CD mechanism.

[4]For examples, see the work of Ayiagari and Gertler (1991), Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998), and Marcet and Singleton (1999).

equilibrium allocations in the limit, as their discount factors go to one, converge to the complete markets allocations. Consequently the pricing implications of the incomplete markets model converge to that of the representative agent model as households become perfectly patient. We provide a qualitatively similar equivalence result that applies only to the risk premium. Our result, however, does not depend on the time discount factor of households. For households with CARA utility, closed form solutions of the individual decision problem in incomplete markets models with idiosyncratic risk are sometimes available, as Willen (1999) shows.[5] In contrast to Willen, we employ CRRA preferences, and we obtain an unambiguous (and negative) result for the impact of uninsurable income risk for the equity premium in case the distribution of individual income shocks is independent of aggregate shocks.

The key ingredients underlying our irrelevance result are (i) a continuum of agents, (ii) CRRA utility, (iii) idiosyncratic labor income risk that is independent of aggregate risk and (iv) solvency constraints or borrowing constraints on total financial wealth that are proportional to aggregate income. We now discuss each of these assumptions in detail to highlight and explain the differences with existing papers in this literature.

First, we need to have a large number of agents in the economy. As forcefully pointed out by Denhaan (2001), in an economy with a finite number of agents, each idiosyncratic shock is by construction an aggregate shock because it changes the wealth distribution and, through these wealth effects, asset prices.

Second, our results rely on the homotheticity of CRRA utility, but not crucially on the time separability of the lifetime utility function; they can easily be extended to Epstein-Zin utility.

Third, in our model labor income grows with the aggregate endowment, as is standard in this literature.[6] In addition, for our results to go through, idiosyncratic income shocks must be distributed independently of aggregate shocks. This explicitly rules out that the variance of idiosyncratic shocks is higher in recessions (henceforth we call such correlation countercyclical cross-sectional variance of labor income shocks, or CCV).

Finally, we can allow households to face either constraints on total net wealth today or state-by-state solvency constraints on the value of their portfolio in each state tomorrow, and these constraints have to be proportional to aggregate income.[7] Our irrelevance result

[5]CARA utility eliminates wealth effects which crucially simplifies the analysis.

[6]By contrast, Weil (1992) derives a positive effect of background risk on the risk premium in a two-period model in which, however, labor income does not have an aggregate endowment component. This non-homotheticity invalidates our mapping from the growing to the stationary economy.

[7]It is more common in the literature to impose short-sale constraints on stocks and bonds separately instead of total financial wealth, but this is done mostly for computational reasons, to bound the state space. In fact, if the solvency constraints are meant to prevent default, they should be directly on total

only survives short-sale constraints on *individual* securities if aggregate consumption growth is uncorrelated over time, as assumed in the benchmark case. In that case households only trade the stock to smooth their consumption. Therefore, in the benchmark case, our result is also robust to transaction costs in the bond market (but not to such costs in the stock market).

As a crucial step in demonstrating our main irrelevance result we prove a theorem that may be of independent theoretical interest. We show that equilibrium allocations and prices in a class of models with idiosyncratic and aggregate risk and incomplete markets can be easily obtained from allocations and interest rates of a stationary equilibrium in a model with only idiosyncratic risk (as in Bewley (1986), Huggett (1993) or Aiyagari (1994)). One first computes a stationary equilibrium for the Bewley model with an appropriate probability measure for idiosyncratic shocks, including a stationary wealth distribution. We then show that scaling up the allocations of the Bewley equilibrium by the aggregate endowment delivers an equilibrium for the model with aggregate uncertainty. The distribution of wealth in that model, normalized by the aggregate endowment, coincides with the stationary wealth distribution of the Bewley equilibrium and thus is (up to the aggregate endowment) stationary as well. Since stationary equilibria in Bewley models are relatively straightforward to compute, our result implies an algorithm for computing equilibria in this class of models which appears to be simpler than the auctioneer algorithm devised by Lucas (1994) and its extension to economies with a continuum of agents. There is also no need for computing a law of motion for the aggregate wealth distribution, or approximating it by a finite number of moments, as in Krusell and Smith (1997, 1998).[8]

This result of our paper makes contact with the literature on aggregation. Constantinides (1982), building on work by Negishi (1960) and Wilson (1968), derives an aggregation result for heterogenous agents in complete market models, implying that assets can be priced off the intertemporal marginal rate of substitution of an agent who consumes the aggregate endowment. Our findings extend his result to a large class of incomplete market models with idiosyncratic income shocks.

The paper is structured as follows. In section 2, we lay out the physical environment of our model. This section also demonstrates how to transform an economy where aggregate income grows stochastically into a stationary economy with a constant aggregate endowment. In section 3 we study this stationary economy, called the Bewley model henceforth. The next

financial wealth (see e.g. Zhang (1996) and Alvarez and Jermann (2000)).

[8]This result also implies the existence of a recursive competitive equilibrium with only asset holdings in the state space, albeit under a transformed probability measure. Kubler and Schmedders (2002) establish the existence of such an equilibrium in more general models, but only under very strong conditions. Miao (2004) relaxes these conditions, but he includes continuation utilities in the state space.

section, section 4, introduces the Arrow model, a model with aggregate uncertainty and a full set of Arrow securities whose payoffs are contingent on the realization of the aggregate shock. We show that a stationary equilibrium of the Bewley model can be mapped into an equilibrium of the Arrow economy just by scaling up allocations by the aggregate endowment. In section 5 we derive the same result for a model where only a one-period risk-free bond can be traded. We call this the *THL* model (for *T*elmer-*H*eaton-*L*ucas model). After briefly discussing the classic *L*ucas-*B*reeden representative agent model (henceforth *LB* model), section 6 shows that risk premia in the representative agent model and the Arrow model (and by implication, in the *THL* model) coincide. Section 7 investigates the robustness of our results with respect to the assumptions about the underlying stochastic income process, and shows in particular that most of our results can be extended to the case where the aggregate shocks are correlated over time and where preferences are not time-separable, but rather follow an Epstein-Zin specification. Finally, section 8 concludes; all proofs are contained in the appendix.

2 Environment

Our exchange economy is populated by a continuum of individuals of measure 1. There is a single nonstorable consumption good. The aggregate endowment of this good is stochastic. Each individual's endowment depends, in addition to the aggregate shock, also on the realization of an idiosyncratic shock. Thus, the economy we study is identical to the one described by Lucas (1994), except that ours is populated by a continuum of agents (as in Bewley (1986), Aiyagari and Gertler (1991), Huggett (1993) and Aiyagari (1994)), instead of just two agents.

2.1 Representation of Uncertainty

We denote the current aggregate shock by $z_t \in Z$ and the current idiosyncratic shock by $y_t \in Y$. For simplicity, both Z and Y are assumed to be finite. Furthermore, let $z^t = (z_0, \ldots, z_t)$ and $y^t = (y_0, \ldots, y_t)$ denote the history of aggregate and idiosyncratic shocks. As shorthand notation, we use $s_t = (y_t, z_t)$ and $s^t = (y^t, z^t)$. We let the economy start at an initial aggregate node z_0. Conditional on an idiosyncratic shock y_0 and thus $s_0 = (y_0, z_0)$, the probability of a history s^t is given by $\pi_t(s^t|s_0)$. We assume that shocks follow a first order Markov process with transition probabilities given by $\pi(s'|s)$.

2.2 Preferences and Endowments

Consumers rank stochastic consumption streams $\{c_t(s^t)\}$ according to the following homothetic utility function:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s^0} \beta^t \pi(s^t|s_0) \frac{c_t(s^t)}{1-\gamma}^{1-\gamma} \tag{1}$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $\beta \in (0,1)$ is the constant time discount factor. We define $U(c)(s^t)$ to be the continuation expected lifetime utility from a consumption allocation $c = \{c_t(s^t)\}$ in node s^t. This utility can be constructed recursively as follows:

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1})$$

where we made use of the Markov property the underlying stochastic processes. The economy's aggregate endowment process $\{e_t\}$ depends only on the aggregate event history; we let $e_t(z^t)$ denote the aggregate endowment at node z^t. Each agent draws a 'labor income' share $\eta(y_t, z_t)$, as a fraction of the aggregate endowment in each period. Her labor income share only depends on the current individual and aggregate event. We denote the resulting individual labor income process by $\{\eta_t\}$, with

$$\eta_t(s^t) = \eta(y_t, z_t) e_t(z^t) \tag{2}$$

where $s^t = (s^{t-1}, y_t, z_t)$. We assume that $\eta(y_t, z_t) > 0$ in all states of the world. The stochastic growth rate of the endowment of the economy is denoted by $\lambda(z^{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$. We assume that aggregate endowment growth only depends on the current aggregate state.

Condition 2.1. *Aggregate endowment growth is a function of the current aggregate shock only:*

$$\lambda(z^{t+1}) = \lambda(z_{t+1})$$

Furthermore, we assume that a Law of Large Numbers holds[9], so that $\pi(s^t|s_0)$ is not only a household's individual probability of receiving income $\eta_t(s^t)$, but also the fraction of the population receiving that income.

In addition to labor income, there is a Lucas tree that yields a constant share α of the total aggregate endowment as capital income, so that the total dividends of the tree are given by $\alpha e_t(z^t)$ in each period. The remaining fraction of the total endowment accrues to

[9] See e.g. Hammond and Sun (2003) for conditions under which a LLN holds with a continuum of random variables.

individuals as labor income, so that $1-\alpha$ denotes the labor income share. Therefore, by construction, the labor share of the aggregate endowment equals the sum over all individual labor income shares:

$$\sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t) = (1-\alpha), \tag{3}$$

for all z_t, where $\Pi_{z_t}(y_t)$ represents the cross-sectional distribution of idiosyncratic shocks y_t, conditional on the aggregate shock z_t. By the law of large numbers, the fraction of agents who draw y in state z only depends on z. An increase in the capital income share α translates into proportionally lower individual labor income shares $\eta(y, z)$ for all (y, z).[10]

At time 0, the agents are endowed with initial wealth θ_0. This wealth represents the value of an agent's share of the Lucas tree producing the dividend flow in units of time 0 consumption, as well as the value of her labor endowment at date 0. We use Θ_0 to denote the initial joint distribution of wealth and idiosyncratic shocks (θ_0, y_0).

Most of our results are derived in a de-trended version of our economy. This de-trended economy features a constant aggregate endowment and a growth-adjusted transition probability matrix. The agents in this de-trended economy, discussed now, have stochastic time discount factors.

2.3 Transformation of Growth Economy into a Stationary Economy

We transform our growing economy into a stationary economy with a stochastic time discount rate and a growth-adjusted probability matrix, following Alvarez and Jermann (2001). First, we define growth deflated consumption allocations (or consumption shares) as

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)} \text{ for all } s^t. \tag{4}$$

Next, we define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \text{ and } \hat{\beta}(s_t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}.$$

Note that $\hat{\pi}$ is a well-defined Markov matrix in that $\sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) = 1$ for all s_t and that $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. For future

[10] Our setup nests the baseline model of Heaton and Lucas (1996), except for the fact that they allow for the capital share α to depend on z.

reference, we also define

$$\hat{\beta}(s^t) = \hat{\beta}(s_0) \cdot \ldots \cdot \hat{\beta}(s_t)$$

and note that $\frac{\hat{\beta}(s^t)}{\hat{\beta}(s^{t-1})} = \hat{\beta}(s_t)$. Finally, let $\hat{U}(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node s^t, under the new transition probabilities and discount factor, defined over consumption shares $\{\hat{c}_t(s^t)\}$

$$\hat{U}(\hat{c})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t)\hat{U}(\hat{c})(s^t, s_{t+1}) \tag{5}$$

In the appendix we prove that this transformation does not alter the agents' ranking of different consumption streams.

Proposition 2.1. *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any* s^t *and any two consumption allocations* c, c'

$$U(c)(s^t) \geq U(c')(s^t) \Longleftrightarrow \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (4).

This result is crucial for demonstrating that equilibrium allocations c for the stochastically growing economy can be found by solving for equilibrium allocations $\hat{c}$ in the transformed economy.

2.4 Independence of Idiosyncratic Shocks from Aggregate Conditions

Next, we assume that idiosyncratic shocks are independent of the aggregate shocks. This assumption is crucial for most of the results in this paper.

Condition 2.2. *Individual endowment shares* $\eta(y_t, z_t)$ *are functions of the current idiosyncratic state* y_t *only, that is* $\eta(y_t, z_t) = \eta(y_t)$*. Also, transition probabilities of the shocks can be decomposed as*

$$\pi(z_{t+1}, y_{t+1}|z_t, y_t) = \varphi(y_{t+1}|y_t)\phi(z_{t+1}|z_t).$$

That is, individual endowment *shares* and the transition probabilities of the idiosyncratic shocks are independent of the aggregate state of the economy z. In this case, the growth-adjusted probability matrix $\hat{\pi}$ and the re-scaled discount factor is obtained by adjusting only the transition probabilities for the *aggregate* shock, ϕ, but not the transition probabilities for the idiosyncratic shocks:

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}|z_t), \text{ and } \hat{\phi}(z_{t+1}|z_t) = \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}.$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state z_t:

$$\hat{\beta}(z_t) = \beta \sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma} \tag{6}$$

The first part of our analysis, until section 6 inclusively, assumes that the aggregate shocks are independent over time:

Condition 2.3. *Aggregate endowment growth is i.i.d.:*

$$\phi(z_{t+1}|z_t) = \phi(z_{t+1}).$$

In this case the growth rate of aggregate endowment is uncorrelated over time, so that the logarithm of the aggregate endowment follows a random walk with drift.[11] As a result, the growth-adjusted discount factor is a constant: $\hat{\beta}(z_t) = \hat{\beta}$, since $\hat{\phi}(z_{t+1}|z_t) = \hat{\phi}(z_{t+1})$. There are two competing effects on the growth-adjusted discount rate: consumption growth itself makes agents more impatient, while the consumption risk makes them more patient.[12]

2.5 A Quartet of Economies

In order to derive our results, we study four models, whose main characteristics are summarized in table 1. The first three models are endowment economies with aggregate shocks. The models differ along two dimensions, namely whether agents can trade a full set of Arrow securities against aggregate shocks, and whether agents face idiosyncratic risk, in addition to aggregate risk. Idiosyncratic risk, if there is any, is never directly insurable.

[11]In section 7 we show that most of our results survive the introduction of persistence in the growth rates if a complete set of contingent claims on aggregate shocks is traded.

[12]The growth-adjusted measure $\hat{\phi}$ is obviously connected to the risk-neutral measure commonly used in asset pricing (see e.g. Harrison and Kreps, 1979). Under our hatted measure, agents can evaluate utils from consumption streams while abstracting from aggregate risk; under a risk-neutral measure, agents can price payoffs by simply discounting at the risk-free rate.

Table 1: Summary of Four Economies			
Model	*Aggregate Shocks*	*Idiosyncr. Shocks*	*Arrow Securities*
THL	Yes	Yes	No
Arrow	Yes	Yes	Yes
BL	Yes	No	Yes
Bewley	No	Yes	N/A

Our primary goal is to understand asset prices in the first model in the table, the *THL* model. This model has idiosyncratic and aggregate risk, as well as incomplete markets. Agents can only insure against idiosyncratic and aggregate shocks by trading a single bond and a stock.

The standard representative agent complete markets Breeden (1979)-Lucas (1978) (*BL*) model lies on the other end of the spectrum; there is no idiosyncratic risk and there is a full menu of Arrow securities for the representative agent to hedge against aggregate risk. Through our analysis we will demonstrate that in the *THL* model the standard representative agent Euler equation for excess returns is satisfied:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}\left(R^s_{t+1}-R_t\right)\right]=0, \tag{7}$$

where R^s_{t+1} is the return on the stock, R_t is the return on the bond and λ_{t+1} is the growth rate of the aggregate endowment. Hence, the aggregate risk premium is identical in the *THL* and the *BL* model. Constantinides (1982) had already shown that, in the case of complete markets, even if agents are heterogeneous in wealth, there exists a representative agent who satisfies the Euler equation for excess returns (7) and also the Euler equation for bonds:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}R_t\right]=1. \tag{8}$$

The key to Constantinides' result is that markets are *complete*. We show that the first Euler equation in (7) survives market incompleteness and potentially binding solvency constraints. The second one does not. To demonstrate this, we employ a third model, the Arrow model (second row in the table). Here, households trade a full set of Arrow securities against aggregate risk, but not against idiosyncratic risk. The fundamental result underlying our asset pricing findings is that equilibria in both the *THL* and the Arrow model can be found by first determining equilibria in a model with *only* idiosyncratic risk (the Bewley model, fourth row in the table) and then by simply scaling consumption allocations in that model by the stochastic aggregate endowment.

We therefore start in section 3 by characterizing equilibria for the Bewley model, a stationary economy with a constant aggregate endowment in which agents trade a single discount bond and a stock.[13] This model merely serves as a computational device. Then we turn to the stochastically growing economy (with different market structures), the one whose asset pricing implications we are interested in, and we show that equilibrium consumption allocations from the Bewley model can be implemented as equilibrium allocations in the stochastically growing *THL* and Arrow model.

3 The Bewley Model

In this model the aggregate endowment is constant at 1. Households face idiosyncratic shocks y that follow a Markov process with transition probabilities $\varphi(y'|y)$. The household's preferences over consumption shares $\{\hat{c}(y^t)\}$ are as defined in equation (5), with the time discount factor $\hat{\beta}$ as defined in equation (6). The adjusted discount factor is $\hat{\beta}$ constant, because the aggregate shocks are i.i.d. (see Condition (2.3)).

3.1 Market Structure

Agents trade only a riskless discount bond and shares in a Lucas tree that yields safe dividends of α in every period. The price of the Lucas tree at time t is denoted by v_t.[14] The riskless bond is in zero net supply. Each household is indexed by an initial condition (θ_0, y_0), where θ_0 denotes its wealth (including period 0 labor income) at time 0.

The household chooses consumption $\{\hat{c}_t(\theta_0, y^t)\}$, bond positions $\{\hat{a}_t(\theta_0, y^t)\}$ and share holdings $\{\hat{\sigma}_t(\theta_0, y^t)\}$ to maximize its normalized expected lifetime utility $\hat{U}(\hat{c})(s^0)$, subject to a standard budget constraint:[15]

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha).$$

Finally, each household faces one of two types of borrowing constraints. The first one restricts household wealth at the end of the current period. The second one restricts household wealth

[13] One of the two assets will be redundant for the households, so that this model is a standard Bewley model, as studied by Bewley (1986), Huggett (1993) or Aiyagari (1994). The presence of both assets will make it easier to demonstrate our equivalence results with respect to the *THL* and Arrow model later on.

[14] The price of the tree is nonstochastic due to the absence of aggregate risk.

[15] We suppress dependence on θ_0 for simplicity whenever there is no room for confusion.

at the beginning of the next period: [16]

$$\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \text{ for all } y^t. \tag{9}$$

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) \geq \hat{M}_t(y^t) \text{ for all } y^t. \tag{10}$$

3.2 Equilibrium in the Bewley Model

The definition of equilibrium in this model is standard.

Definition 3.1. *For an initial distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the Bewley model consists of trading strategies $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$, and prices $\{\hat{R}_t, \hat{v}_t\}$ such that*

1. *Given prices, trading strategies solve the household maximization problem*

2. *The goods markets and asset markets clear in all periods t*

$$\begin{aligned}
\int \sum_{y^t} \varphi(y^t|y_0)\hat{c}_t(\theta_0, y^t) d\Theta_0 &= 1. \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t) d\Theta_0 &= 0. \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t) d\Theta_0 &= 1.
\end{aligned}$$

In the absence of aggregate risk, the bond and the stock are perfect substitutes for households, and no-arbitrage implies the following restriction on equilibrium stock prices and interest rates:

$$\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}.$$

In addition, with these equilibrium prices household portfolios are indeterminate. Without loss of generality one can therefore focus on trading strategies in which households only trade the stock, but not the bond: $\hat{a}_t(\theta_0, y^t) \equiv 0$.[17]

A *stationary equilibrium* in the Bewley model consists of a constant interest rate $\hat{R}$, a share price $\hat{v}$, optimal household allocations and a time-invariant measure Φ over income

[16] This distinction is redundant in the Bewley model, but it will become meaningful in our models with aggregate risk.

[17] Alternatively, we could have agents simply trade in the bond and adjust the net supply of bonds to account for the positive capital income α in the aggregate. We only introduce both assets into the Bewley economy to make the mapping to allocations in the Arrow and *THL* models simpler.

shocks and financial wealth.[18] In the stationary equilibrium households move within the invariant wealth distribution, but the wealth distribution itself is constant over time.

4 The Arrow Model

We now turn to our main object of interest, the economy with aggregate risk. We first consider the Arrow market structure in which households can trade shares of the stock and a complete menu of contingent claims on aggregate shocks. Idiosyncratic shocks are still uninsurable. We demonstrate in this section that the allocations and prices of a stationary Bewley equilibrium can be made into equilibrium allocations and prices in the Arrow model with aggregate risk.

4.1 Trading

Let $a_t(s^t, z_{t+1})$ denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is z_{t+1}, irrespective of the idiosyncratic shock y_{t+1}. Its price today is given by $q_t(z^t, z_{t+1})$. In addition, households trade shares in the Lucas tree. We use $\sigma_t(s^t)$ to denote the number of shares a household with history $s^t = (y^t, z^t)$ purchases today and we let $v_t(z^t)$ denote the price of one share.

An agent starting period t with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \leq \theta_t(s^t). \tag{11}$$

If next period's state is $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$, her wealth is given by her labor income, the payoff from the contingent claim purchased in the previous period as well as the value of her position on the stock, including dividends:

$$\theta_{t+1}(s^{t+1}) = \eta(y_{t+1}, z_{t+1}) e_{t+1}(z_{t+1}) + a_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}). \right] \tag{12}$$

In addition to the budget constraints, the households' trading strategies are subject to solvency constraints of one of two types. The first type of constraint imposes a lower bound

[18] See Chapter 17 of Ljungqvist and Sargent (2004) for the standard formal definition and the straightforward algorithm to compute such a stationary equilibrium.

on the value of the asset portfolio at the end of the period today,

$$\sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq K_t(s^t), \tag{13}$$

while the second type imposes state-by-state lower bounds on net wealth tomorrow,

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}) \right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \tag{14}$$

We assume these solvency constraints are at least tight enough to prevent Ponzi schemes. In addition, we impose restrictions on the solvency constraints that make them proportional to the aggregate endowment in the economy:

Condition 4.1. *We assume that the borrowing constraints only depend on the aggregate history through the level of the aggregate endowment. That is, we assume*

$$K_t(y^t, z^t) = \hat{K}_t(y^t) e_t(z^t),$$

and

$$M_t(y^t, z^t, z_{t+1}) = \hat{M}_t(y^t) e_{t+1}(z^{t+1}).$$

If the constraints did not have this feature in a stochastically growing economy, the constraints would become more or less binding as the economy grows, clearly not a desirable feature[19]. The definition of an equilibrium is completely standard (see section A.1 of the Appendix).

Instead of working with the model with aggregate risk, we transform the Arrow model into a stationary model. As we are about to show, the equilibrium allocations and prices in the de-trended model are the same as the allocations and prices in a stationary Bewley equilibrium.

4.2 Equilibrium in the De-trended Arrow Model

We use hatted variables to denote the variables in the stationary model. Households rank consumption shares $\{\hat{c}_t\}$ in exactly the same way as original consumption streams $\{c_t\}$. Dividing the budget constraint (11) by $e_t(z^t)$ and using equation (12) yields the deflated

[19] In the incomplete markets literature the borrowing constraints usually have this feature (see e.g. Heaton and Lucas (1996)). It is easy to show that solvency constraints that are not too tight in the sense of Alvarez and Jermann (2000) satisfy this condition.

budget constraint:

$$\begin{aligned} & \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \\ \leq \; & \eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha \right], \end{aligned} \tag{15}$$

where we have defined the deflated Arrow positions as $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and prices as $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1}) \lambda(z_{t+1})$. The deflated stock prices are given by $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$. Similarly, by deflating the solvency constraints (13) and (14), using condition (4.1), yields:

$$\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) \;\geq\; \hat{K}_t(y^t). \tag{16}$$

$$\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[\hat{v}_{t+1}(z^{t+1}) + \alpha \right] \;\geq\; \hat{M}_t(y^t) \text{ for all } z_{t+1}. \tag{17}$$

Finally, the goods market clearing condition is given by[20]

$$\int \sum_{y^t} \pi(y^t | y_0) \hat{c}_t(\theta_0, s^t) d\Theta_0 = 1. \tag{18}$$

The asset market clearing conditions are exactly the same as before. In the stationary economy, the household maximizes $\hat{U}(\hat{c})(s_0)$ by choosing consumption, Arrow securities and shares of the Lucas tree, subject to the budget constraint (50) and the solvency constraint (16) or (17) in each node s^t. The definition of a competitive equilibrium in the de-trended Arrow economy is straightforward.

Definition 4.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the de-trended Arrow model consists of trading strategies $\{\hat{a}_t(\theta_0, s^t, z_{t+1})\}$, $\{\hat{\sigma}_t(\theta_0, s^t)\}$, $\{\hat{c}_t(\theta_0, s^t)\}$ and prices $\{\hat{q}_t(z^t, z_{t+1})\}$, $\{\hat{v}_t(z^t)\}$ such that*

1. *Given prices, trading strategies solve the household maximization problem*

2. *The goods market clears, that is, equation* (18) *holds for all z^t.*

3. *The asset markets clear*

$$\begin{aligned} \int \sum_{y^t} \varphi(y^t | y_0) \hat{\sigma}_t(\theta_0, s^t) d\Theta_0 &= 1 \\ \int \sum_{y^t} \varphi(y^t | y_0) \hat{a}_t(\theta_0, s^t, z_{t+1}) d\Theta_0 &= 0 \text{ for all } z_{t+1} \in Z \end{aligned}$$

[20]The conditional probabilities simplify due to condition (2.2).

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are listed in the appendix in section (A.1.1). These are necessary and sufficient conditions for optimality on the household side. Now we are ready to establish the equivalence between equilibria in the Bewley model and in the Arrow model.

4.3 Equivalence Results

We now show that equilibria in the Bewely model can be mapped into equilibria of the stochastically growing Arrow model.

Theorem 4.1. *An equilibrium of the Bewley model* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t),\ \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow model with growth,* $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}) \\ v_t(z^t) &= \hat{v}_t e_t(z^t) \\ q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} \end{aligned} \tag{19}$$

The proof is given in the appendix, but here we provide its main intuition. Conjecture that equilibrium prices of Arrow securities in the de-trended Arrow model are given by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}. \tag{20}$$

The Euler equation for an unconstrained household with respect to Arrow securities reads as (see section (A.1.1) in the appendix)

$$1 = \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))}.$$

But under the maintained assumptions 2.2 and 2.3 and under the conjecture that consumption allocations in the de-trended Arrow model only depend on idiosyncratic shock histories

y^t and not on $s^t = (y^t, z^t)$ this Euler equation reduces to

$$
\begin{aligned}
1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} && (21)\\
&= \hat{\beta}\hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}, && (22)
\end{aligned}
$$

where we used the conjectured form of prices in (20). But this is exactly the Euler equation with respect to bonds in the Bewley model. Since Bewley equilibrium consumption allocations satisfy this condition, they therefore satisfy the Euler equation in the de-trended Arrow model, if prices are of the form (20). The proof in the appendix shows that a similar argument applies for the Euler equation with respect to the stock (under the conjectured stock prices), and also shows that for agents whose solvency constraints binds the Lagrange multipliers on the constraints in the Bewley equilibrium are also valid Lagrange multipliers for the constraints in the de-trended Arrow model. This implies, in particular, that our results go through independent of how tight the solvency constraints are. Once one has established that allocations and prices of a Bewley equilibrium are an equilibrium in the de-trended Arrow model, one simply needs to scale up allocation and prices by the appropriate growth factors to obtain the equilibrium prices and allocations in the stochastically growing Arrow model, as stated in the theorem.

It is straightforward to compute risk-free interest rates for the Arrow model. By summing over aggregate states tomorrow on both sides of equation (20), we obtain that the risk-free rate in the de-trended Arrow model coincides with that of the Bewley model:

$$
\hat{R}_t^A = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \hat{R}_t. \tag{23}
$$

Once we have determined risk free interest rates for the de-trended economy, $\hat{R}_t^A = \hat{R}_t$, we can back out the implied interest rate for the original growing Arrow economy, using (19) in the previous theorem.[21]

Corollary 4.1. *If equilibrium risk-free interest rates in the de-trended Arrow model are given by* (23)*, equilibrium risk-free interest rates in the Arrow model with aggregate risk are given by*

$$
R_t^A = \frac{1}{\sum_{z_{t+1}} q_t(z^t, z_{t+1})} = \hat{R}_t \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}. \tag{24}
$$

[21]The dependence of $\hat{R}_t^A$ on time t is not surprising since, for an arbitrary initial distribution of assets Θ_0, we cannot expect the equilibrium to be stationary. In the same way we expect that $\hat{v}_t(z^t)$ is only a function of t as well, but not of z^t.

This result implies that, in the absence of aggregate risk (λ is only a function of time, but not of z_{t+1}) the risk-free rates in the original and deflated economy are related by the familiar relation $R_t^A = \hat{R}_t^A \lambda_{t+1}$where λ_{t+1} is the gross growth rate of endowment between period t and $t+1$.

The theorem implies that we can solve for an equilibrium in the Bewley model of section 3 (and in, in particular, a stationary equilibrium), including risk free interest rates $\hat{R}_t$, and we can deduce the equilibrium allocations and prices for the Arrow economy from those in the Bewley economy, using the mapping described in theorem 4.1. As shown, the key to this result is that households in the Bewley model face exactly the same Euler equations as the households in the de-trended version of the Arrow model.

This theorem has several important implications. First, we will use this equivalence result to show below that asset prices in the Arrow economy are identical to those in the representative agent economy, except for a lower risk-free interest rate (and a higher price/dividend ratio for stocks).[22] Second, the existence proofs in the literature for stationary equilibria in the Bewley model directly carry over to the stochastically growing economy[23]. Third, the moments of the wealth distribution vary over time but proportionally to the aggregate endowment. If the initial wealth distribution in the de-trended model corresponds to an invariant distribution in the Bewley model, then for example the ratio of the mean to the standard deviation of the wealth distribution is constant in the Arrow model with aggregate risk as well. Finally, an important result of the previous theorem is that, in the Arrow equilibrium, the trade of Arrow securities is simply proportional to the aggregate endowment: $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1})$, or, equivalently, in the de-trended Arrow model households choose not to make their contingent claims purchases contingent on next period's aggregate shock: $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$. Furthermore, since in the Bewley model without loss of generality $\hat{a}_t(\theta_0, y^t) = 0$, we can focus on the situation where Arrow securities are not traded at all: $a_t(\theta_0, s^t, z_{t+1}) = 0$. This no-trade result for contingent claims suggests that the equivalence will carry over to economies with more limited asset structures. That is what we show in the next section.

[22]The fact that the risk-free interest rate is lower comes directly from the fact that interest rates in the Bewley model are lower than in the corresponding representative agent model without aggregate risk.

[23]See e.g. Huggett (1993), Aiyagari (1994) or Miao (2002) for (elements of) existence proofs. Uniqueness of a stationary equilibrium is much harder to establish. Our equivalence result shows that for any stationary Bewley equilibrium there exists a corresponding Arrow equilibrium in the model with aggregate risk. Furthermore note that our result does not rule out other Arrow equilibria either.

5 The *THL* Model

We now turn our attention to the main model of interest, namely the model with a stock and a single uncontingent bond. This section establishes the equivalence of equilibria in the *THL* model and the Bewley model by showing that the optimality conditions in the de-trended Arrow and the de-trended *THL* model are identical. In addition, we show that in the benchmark case with i.i.d. aggregate endowment growth shocks agents do not even trade bonds in equilibrium.

5.1 Market Structure

In the *THL* economy, agents only trade a one-period discount bond and a stock. An agent who starts period t with initial wealth composed on bond and stock payout and labor income buys consumption commodities in the spot market and trades a one-period bond and the stock, subject to the budget constraint:

$$c_t(s^t) + \frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \leq \eta(y_t)e_t(z_t) + b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1})\left[v_t(z^t) + \alpha e_t(z_t)\right] \quad (25)$$

Here, $b_t(s^t)$ denotes the amount of bonds purchased and $R_t(z^t)$ is the gross interest rate from period t to $t+1$. As was the case in the Arrow model, short-sales of the bond and the stock are constrained by a lower bound on the value of the portfolio today,

$$\frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t), \quad (26)$$

or a state-by-state constraint on the value of the portfolio tomorrow,

$$b_t(s^t) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \quad (27)$$

Since $b_t(s^t)$ and $\sigma_t(s^t)$ are chosen before z_{t+1} is realized, at most one of the constraints (27) will be binding at a given time. The definition of an equilibrium for the *THL* model follows directly. (see section (A.2) in the appendix). We now show that the equilibria in the Arrow and the *THL* model coincide. As a corollary, it follows that the asset pricing implications of both models are identical. In order to do so, we first transform the model with growth into a stationary, de-trended model.

5.2 Equilibrium in the De-trended *THL* Model

Dividing the budget constraint (25) by $e_t(z^t)$ we obtain

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right],$$

where we define the deflated bond position as $\hat{b}_t(s^t) = \frac{b_t(s^t)}{e_t(z^t)}$. Using condition (4.1), the solvency constraints in the de-trended economy are simply:

$$\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \quad \geq \quad \hat{K}_t(y^t), \text{ or} \tag{28}$$

$$\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] \quad \geq \quad \hat{M}_t(y^t) \text{ for all } z_{t+1}. \tag{29}$$

The definition of equilibrium in the de-trended *THL* model is straightforward and hence omitted.[24] We now show that equilibrium consumption allocations in the de-trended *THL* model coincide with those of the Arrow model. For simplicity, we first abstract from binding borrowing constraints and then extend our results to that case later on.

5.3 Equivalence Results

As for the Arrow economy, we can show that the Bewley equilibrium allocations and prices constitute, after appropriate scaling by endowment (growth) factors, an equilibrium of the *THL* model with growth.

Theorem 5.1. *An equilibrium of a stationary Bewley economy, given by trading strategies* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and prices* $\{\hat{R}_t, \hat{v}_t\}$*, can be made into an equilibrium for the* THL *model with growth,* $\{b_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$, $\left\{\sigma_t^{THL}(\theta_0, s^t)\right\}$ *and* $\{R_t(z^t)\}$ *and* $\{v_t(z^t)\}$ *where*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t)e_t(z^t) \\ \sigma_t^{THL}(\theta_0, s^t) &= \frac{\hat{a}_t(\theta_0, y^t)}{[\hat{v}_{t+1} + \alpha]} + \hat{\sigma}_t(\theta_0, y^t) \\ v_t(z^t) &= \hat{v}_t e_t(z^t) \\ R_t(z^t) &= \hat{R}_t * \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}} \end{aligned}$$

[24] We list the first order conditions for household optimality and the transversality conditions in section (A.2) of the appendix.

and bond holdings given by $b_t(\theta_0, s^t) = 0$.

The crucial step of the proof, given in the appendix, shows that Bewley allocations, given the prices proposed in the theorem, satisfy the necessary and sufficient conditions for household optimality and all market clearing conditions in the de-trended *THL* model.

This theorem again has several important consequences. First, equilibrium risk-free rates in the Arrow and in the *THL* model coincide, despite the fact that the set of assets agents can trade to insure consumption risk differs in the two models. Second, in equilibrium of the *THL* model, the bond market is inoperative: $b_{t-1}(s^{t-1}) = \hat{b}_{t-1}(s^{t-1}) = 0$ for all s^{t-1}. Therefore all consumption smoothing is done by trading stocks, and agents keep their net wealth proportional to the level of the aggregate endowment.[25]

In summary, our results show that one can solve for equilibria in a standard Bewley model and then map this equilibrium into an equilibrium for both the Arrow model and the *THL* model *with* aggregate risk. The risk-free interest rate and the price of the Lucas tree coincide in the stochastic Arrow and *THL* economies. Finally, without loss of generality, we can restrict attention to equilibria in which bonds are not traded; consequently transaction costs in the bond market would not change our results. Transaction costs in the stock market of course would (see section (7)). In addition, this implies that our result is robust to the introduction of short-sale constraints imposed on stocks and bonds separately, because agents choose not to trade bonds in equilibrium.

In both the Arrow and the *THL* model, households do not have a motive for trading bonds, unless there are short-sale constraints on stocks. We do not deal with this case. In addition, the no-trade result depends critically on the i.i.d assumption for aggregate shocks, as we will show in section (7). If the aggregate shocks are not i.i.d, agents want to hedge against the implied shocks to interest rates. We will show in section (7) that these interest rate shocks look like taste shocks in the de-trended model.

But first we compare the asset pricing implications of the equilibria just described in the Arrow and the *THL* models to those emerging from the *BL* (standard representative agent) model.

[25]There is a subtle difference between this result and the corresponding result for the Arrow model. In the Arrow model we demonstrated that contingent claims positions were in fact uncontingent: $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ and equal to the Bond position in the Bewley equilibrium, but not necessarily equal to zero. In the *THL* model bond positions *have to be* zero. But since bonds in the Bewley equilibrium are a redundant asset, one can restrict attention to the situation where $\hat{a}_t(\theta_0, y^t) = 0$, although this is not necessary for our results.

6 Asset Pricing Implications

This section shows that the multiplicative risk premium on a claim to aggregate consumption in the *THL* model -and the Arrow model- equals the risk premium in the representative agent model. Uninsurable idiosyncratic income risk only lowers the risk-free rate.

6.1 Consumption-CAPM

The benchmark model of consumption-based asset pricing is the representative agent *BL* model. The representative agent owns a claim to the aggregate 'labor' income stream $\{(1-\alpha)e_t(z^t)\}$ and she can trade a stock (a claim to the dividends $\alpha e_t(z^t)$ of the Lucas tree), a bond and a complete set of Arrow securities.[26]

First, we show that the *B*reeden-*L*ucas Consumption-CAPM also prices excess returns on the stock in the *THL* model and the Arrow model. Let R^s denote the return on a claim to aggregate consumption. We have

Lemma 6.1. *The* BL *Consumption-CAPM prices excess returns in the Arrow model and the* THL *model: In equilibrium in both models*

$$E_t\left[\left(R^s_{t+1}-R_t\right)\beta\left(\lambda_{t+1}\right)^{-\gamma}\right]=0$$

This result follows directly from the Euler equation in (22). It has important implications for empirical work in asset pricing. First, and conditional on either the Arrow model or the *THL* model being the correct model of the economy, despite the existence of market incompleteness and binding solvency constraints an econometrician can estimate the coefficient of risk aversion (or the intertemporal elasticity of substitution) directly from aggregate consumption data and the excess return on stocks, as in Hansen and Singleton (1982). Second the result provides a strong justification for explaining the cross-section of excess returns using the CCAPM without trying to match the risk-free rate. The implications of the *BL,* the Arrow and the *THL* model are the same with respect to excess returns, while not with respect to the risk-free rate.

6.2 Risk Premia

We now show that, perhaps not surprisingly after having seen the previous result, the equilibrium risk premium in the Arrow and the *THL* model is identical to the one in the representative agent model.[27] While the risk-free rate is higher in the representative agent model

[26] See section (A.3) in the Appendix for a complete description.

[27] Note that this does *not* immediately follow from the result in Lemma 6.1.

than in the Arrow and *THL* model, and consequently the price of the stock is correspondingly lower, the multiplicative risk premium is the same in all three models and it is constant across states of the world.

In order to demonstrate our main result we first show that the stochastic discount factors that price stochastic payoffs in the representative agent model and the Arrow (and thus the *THL*) model only differ by a non-random multiplicative term, equal to the ratio of (growth-deflated) risk-free interest rates in the two models. In what follows we use the superscript RE to denote equilibrium entities in the representative agent model.

Proposition 6.1. *The equilibrium stochastic discount factor in the Arrow and the* THL *model given by*

$$m_{t+1}^{A} = m_{t+1}^{RE}\kappa_t$$

where the non-random multiplicative term is given

$$\kappa_t = \frac{\hat{R}_t^{RE}}{\hat{R}_t} \geq 1$$

and m_{t+1}^{RE} is the stochastic discount factor in the representative agent model.

Note that the term κ_t is straightforward to compute as it only involves the equilibrium risk-free interest rates from the stationary version of the representative agent model, $\hat{R}_t^{RE}$, and the equilibrium interest rates from the Bewley model, $\hat{R}_t$.

The proof that risk premia are identical in the representative agent model and the Arrow as well as the *THL* model now follows from the previous decomposition of the stochastic discount factor.[28] Let $R_{t,j}\left[\{d_{t+k}\}\right]$ denote the j-period holding return on a claim to the endowment stream $\{d_{t+k}\}_{k=0}^{\infty}$ at time t. Consequently $R_{t,1}\left[1\right]$ is the gross risk-free rate and $R_{t,1}\left[\alpha e_{t+k}\right]$ is the one-period holding return on a k-period strip of the aggregate endowment (a claim to α times the aggregate endowment k periods from now). Thus $R_{t,1}\left[\{\alpha e_{t+k}\}\right]$ is the one period holding return on an asset (such as a stock) that pays α times the aggregate endowment in all future periods. Finally define the multiplicative risk premium as the ratio of the expected return on stocks and the risk-free rate:

$$1 + \nu_t = \frac{E_t R_{t,1}\left[\{\alpha e_{t+k}\}\right]}{R_{t,1}\left[1\right]}$$

With this notation in place, we can state now our main result.

[28]The proof strategy follows Alvarez and Jermann (2001) who derive a similar result in the context of a *complete markets* model populated by two agents that face endogenous solvency constraints.

Theorem 6.1. *The multiplicative risk premium in the Arrow model and* THL *model equals that in the representative agent model*

$$1 + \nu_t^A = 1 + \nu_t^{\text{THL}} = 1 + \nu_t^{RE}$$

and is constant across states of the world.

Thus the extent to which households smooth idiosyncratic income shocks in the Arrow model or in the *THL* model has absolutely no effect on the size of risk premia; it merely lowers the risk-free rate. Luttmer (1991) and Cochrane and Hansen (1992) had already established a similar aggregation result for the case in which households face market wealth constraints, but in a complete markets environment. We show that their result survives even if households trade only a stock and a bond. Market incompleteness does not generate any dynamics in the conditional risk premia either: the conditional risk premium is constant.

7 Robustness and Extensions of the Main Results

In this section we investigate how robust our results are to the assumption that aggregate shocks are *i.i.d* over time, which implies that the growth rate of the aggregate endowment is *i.i.d* over time. We show that our results go through even aggregate shocks are serially correlated, but only for the Arrow model and only if solvency constraints are either never binding or are chosen appropriately.

7.1 Non-iid Aggregate Shocks

Assume that the aggregate shock z follows a first order Markov chain characterized by the transition matrix $\phi(z'|z) > 0$. So far we studied the special case in which $\phi(z'|z) = \phi(z')$. Recall that the growth-adjusted Markov transition matrix and time discount factor are given by

$$\hat{\phi}(z'|z) = \frac{\phi(z'|z)\lambda(z')^{1-\gamma}}{\sum_{z'} \phi(z'|z)\lambda(z')^{1-\gamma}} \text{ and } \hat{\beta}(z) = \beta \sum_{z'} \phi(z'|z)\lambda(z')^{1-\gamma}.$$

Thus if ϕ is serially correlated, so is $\hat{\phi}$, and the discount factor $\hat{\beta}$ does depend on the current aggregate state of the world. This indicates (and will show this below) that the aggregate endowment shock acts as an aggregate taste shock in the de-trended economy which renders all households more or less impatient. Note that since this shock affects all households in the same way they will not able to insure against it. Thus this shock affects the price/dividend ratio and the interest rate but leaves the risk premium unaltered. In contrast to our previous results, however, now there is trade in Arrow securities in equilibrium, so the equivalence between equilibria in the Arrow and the *THL* model breaks down.

7.1.1 Stationary Bewley Model

Following our previous strategy we will argue that equilibrium allocations and prices from a stationary version of the model, again called the Bewley model, can be implemented, after appropriate scaling by the aggregate endowment, as equilibria in the stochastically growing model. Since now discount factors are subject to aggregate shock we first have to choose an appropriate discount factor for the Bewley model. We will choose a process of discount factors that assures that Bewley equilibrium allocations satisfy the time zero budget constraint in the model with aggregate shocks when the initial wealth distribution Θ_0 in the two models coincide.

Let

$$\hat{\beta}_{0,\tau}(z^\tau|z_0) = \hat{\beta}(z_0)\hat{\beta}(z_1)\dots\hat{\beta}(z_\tau)$$

denote the time discount factor between period 0 and period $\tau+1$, given by the product of one-period time discount factors. Define the average (across aggregate shocks) time discount factor between period 0 and t as

$$\widetilde{\beta}_t = \sum_{z^{t-1}|z_0} \widehat{\phi}(z^{t-1}|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0), t \geq 1, \tag{30}$$

where $\widehat{\phi}(z^{t-1}|z_0)$ is the probability distribution over z^{t-1} induced by $\phi(z'|z)$. If aggregate shocks are *i.i.d*, then we have that $\widetilde{\beta}_t = \hat{\beta}^t$, as before. Note that since z_0 is a fixed initial condition we chose not to index $\widetilde{\beta}_t$ by z_0 to make sure it is understood that $\widetilde{\beta}_t$ is nonstochastic.

In order to construct equilibrium allocations in the stochastically growing model we now show that equilibrium allocations and interest rates in the Bewley model with a sequence of time discount factors $\left\{\widetilde{\beta}_t\right\}_{t=1}^{\infty}$ can be implemented as equilibrium allocations and interest rates for the *actual* Arrow model with stochastic discount factors.The crucial adjustment in this mapping is to scale the risk-free interest rate in proportion to the taste shock $\hat{\beta}(z)$.

To understand the effect of these aggregate taste shocks on the time discount rate to be used in the Bewley model consider the following simple, purely illustrative example.

Example 7.1. *Suppose that $\hat{\beta}(z) = e^{-\hat{\rho}(z)}$ is lognormal and i.i.d*[29] *where $\hat{\rho}(z)$ has mean $\hat{\rho}$ variance σ^2. Define the average t-period time discount rate $\widetilde{\rho}_t$ by $\widetilde{\beta}_t = e^{-\widetilde{\rho}_t}$. Then the average*

[29] Of course, if z follows the Markov chain with transition probabilities $\hat{\phi}$ then $\hat{\beta}$ will neither be *i.i.d* nor lognormal.

one-period discount rate used in the Bewley model is given by:

$$\frac{\widetilde{\rho}_t}{t} = \widehat{\rho} - \frac{1}{2}\sigma^2 \text{ for any } t \geq 1$$

Thus the presence of taste shocks ($\sigma^2 > 0$) in the de-trended Arrow model induces a discount rate $\widetilde{\rho}$ to be used in the Bewley model that is lower than the mean discount rate $\hat{\rho}$ because of the risk associated with the taste shocks. This suggests lower associated risk-free interest rates than in the absence of taste shocks (which originate from aggregate endowment shocks in the stochastically growing model).

As before denote by $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\left\{\hat{R}_t, \hat{v}_t\right\}$ Bewley equilibrium allocations and prices, for a given sequence of time discount factors $\left\{\widetilde{\beta}_t\right\}$. Since, again as before, stocks and bonds are both risk free assets and thus perfect substitutes in the Bewley model, prices satisfy the no-arbitrage condition

$$\hat{R}_t = \frac{\hat{v}_{+1} + \alpha}{\hat{v}_t}$$

and only total wealth positions in the Bewley economy are uniquely pinned down. Without loss of generality, we focus on the case where $\hat{a}_t(\theta_0, y^t) = 0$ for all y^t. We now argue that the allocation $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ can be made into an Arrow equilibrium, and in the process show why we need to choose the specific discount factor sequence in (30) for the Bewley model.

To fix notation that will be useful below let

$$\tilde{Q}_{t,\tau} = \prod_{j=0}^{\tau-t-1} \hat{R}_{t+j}^{-1} = \frac{1}{\hat{R}_{t,\tau}} \tag{31}$$

denote the Bewley equilibrium price of one unit of consumption to be delivered at time τ, in terms of consumption at time t. By convention $\tilde{Q}_\tau = \tilde{Q}_{0,\tau}$ and $\tilde{Q}_{\tau,\tau} = 1$ for all τ. Here is the gross risk-free interest rate between period t and τ in the Bewley equilibrium.

7.1.2 Arrow Model

In contrast to the Bewley model the de-trended Arrow model features aggregate shocks to the time discount factor $\hat{\beta}$. These need to be reflected in prices. We therefore propose equilibrium prices for the de-trended Arrow model, then show that with these prices the Bewley equilibrium allocations, in turn, satisfy the Euler equations and the intertemporal budget constraint in the de-trended Arrow model. This implies that, absent binding solvency constraints, the Bewley equilibrium can be made into an equilibrium of the de-trended Arrow model, and thus, after the appropriate scaling, into an equilibrium of the original Arrow model. Finally we discuss potentially binding solvency constraints and the *THL* model.

We conjecture that Arrow-Debreu prices in the deflated Arrow model are given by

$$\hat{Q}_t(z^t|z_0) = \hat{\phi}(z^t|z_0)\tilde{Q}_t \frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\widetilde{\beta}_t} = \hat{\phi}(z^t|z_0)\frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\widetilde{\beta}_t \hat{R}_{0,t}} \tag{32}$$

where $\tilde{Q}_t$ was defined above as the time 0 price of consumption in period t in the Bewley model. Prices of Arrow securities are then given by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{Q}_{t+1}(z^{t+1}|z_0)}{\hat{Q}_t(z^t|z_0)} = \hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)\frac{1}{\hat{R}_t}\frac{\widetilde{\beta}_t}{\widetilde{\beta}_{t+1}} = \hat{q}_t(z_{t+1}|z_t) \tag{33}$$

where we used the fact that $\frac{1}{\hat{R}_t} = \frac{\hat{R}_{0,t}}{\hat{R}_{0,t+1}}$. Note that Arrow prices are Markovian, since $\hat{R}_t$ and $\widetilde{\beta}_t, \widetilde{\beta}_{t+1}$ are deterministic. Equation (33) implies that interest rates in the de-trended Arrow model are given by

$$\hat{R}^A_t(z_t) \equiv \sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t) = \hat{R}_t \frac{\widetilde{\beta}_{t+1}}{\hat{\beta}(z_t)\widetilde{\beta}_t}. \tag{34}$$

Thus interest rates now depend on the current aggregate state of the world z_t. Consistently, we conjecture asset prices in the de-trended Arrow model to satisfy

$$\hat{v}_t(z^t) = \hat{v}_t(z_t) = \sum_{z_{t+1}} \hat{\phi}(z_{t+1}|z_t)\left(\frac{\hat{v}_{t+1}(z_{t+1}) + \alpha}{\hat{R}^A_t(z_t)}\right) \tag{35}$$

Armed with these conjectured prices we can now prove the following

Lemma 7.1. *Absent solvency constraints, the household Euler equations are satisfied in the Arrow model at the Bewley allocations* $\{\hat{c}_{t+1}(y^t, y_{t+1})\}$ *and Arrow prices* $\{\hat{q}_t(z_{t+1}|z_t)\}$ *given by (33).*

Trading We now investigate what asset trades support Bewley equilibrium consumption allocations in the de-trended Arrow model and show that the implied contingent claims positions clear the market for Arrow securities.

At any point in time and any node of the event tree, the position of Arrow securities at the beginning of the period, plus the value of the stock position cum dividends, has to finance the value of excess demand from today into the infinite future. Thus the Arrow securities position implied by the Bewley equilibrium allocation $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ is given by[30]

$$\begin{aligned}\hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) &= \hat{c}_t(\theta_0, y^t) - \eta(y_t) + \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_\tau(z^\tau|z_t)\varphi(y^\tau|y^t)\left(\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right) \\ &\quad -\hat{\sigma}_{t-1}(\theta_0, y^{t-1})\left[\hat{v}_t(z_t) + \alpha\right] \\ &= \hat{a}_{t-1}(\theta_0, y^{t-1}, z_t). \end{aligned} \tag{36}$$

Proposition 7.1. *The contingent claims positions implied by the Bewley allocations in* (36) *clear the Arrow securities markets, that is*

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0)\hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 = 0 \text{ for all } z^t.$$

Using the fact that wealth from stock holdings at the beginning of the period

$$\hat{\sigma}_{t-1}(\theta_0, y^{t-1})\left[\hat{v}_t + \alpha\right] = \alpha\hat{\sigma}_{t-1}(\theta_0, y^{t-1}) \sum_{\tau=t}^{\infty} \widetilde{Q}_{t,\tau}$$

has to finance future excess consumption demand also in the Bewley equilibrium we can restate the contingent claims positions as

$$\begin{aligned}\hat{a}_{t-1}(\theta_0, y^{t-1}, z_t) &= \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_{t,\tau}\right) \sum_{y^\tau} \varphi(y^\tau|y^t)\left(\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right) \\ &\quad -\hat{\sigma}_{t-1}(\theta_0, y^{t-1})\alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_{t,\tau}\right)\end{aligned}$$

This makes explicit that the Arrow securities positions held by households are used to hedge against the interest rate shocks that govern the difference between the stochastic $\hat{Q}_\tau(z^\tau|z_t)$ and the deterministic $\widetilde{Q}_{t,\tau}$. If aggregate endowment growth is i.i.d, taste shocks in the Arrow model are absent, and from (34) we see that the interest rates are deterministic. The gap between $\hat{Q}_\tau(z^\tau|z_t)$ and $\widetilde{Q}_{t,\tau}$ is zero and no Arrow securities are traded in equilibrium, confirming the results in section 4.

In order to close our argument that a Bewley equilibrium can be implemented as equilibrium in the de-trended Arrow model (and thus the original Arrow model) we need to show that no initial wealth transfers between individuals are required for this implementation. In other words, we need to make sure that the initial Arrow securities position $\hat{a}_{-1}(\theta_0, y^{-1}, z_0)$

[30] We will verify below that the price of the stock in the de-trended Arrow model satisfies $\hat{v}_t(z^t) = \hat{v}_t(z_t)$.

implied by (36) at time 0, is zero for all households.[31]

To do so we proceed in two steps. First, we show that the average time zero state prices in the Arrow model coincide with the state prices in the Bewley model. For this result to hold our particular choice of time discount factors $\left\{\widetilde{\beta}_t\right\}$ for the Bewley model is crucial.

Lemma 7.2. *The conjectured prices for the Arrow model in* (32) *and the prices in the Bewley model defined in* (31) *satisfy*

$$\sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_0) = \tilde{Q}_\tau$$

Finally, using this result we can establish that no wealth transfers are necessary to implement the Bewley equilibrium as equilibrium in the de-trended Arrow model.

Lemma 7.3. *The Arrow securities position at time 0 given in* (36) *is zero:*

$$\hat{a}_{-1}(\theta_0, y^{-1}, z_0) = 0.$$

Having established that the Bewley equilibrium is an equilibrium for the de-trended Arrow model with the same initial wealth distribution Θ_0, the following theorem obviously results.

Theorem 7.2. *An equilibrium of the Bewley model* $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *where households have a sequence of time discount factors* $\left\{\widetilde{\beta}_t\right\}$ *can be made into an equilibrium for the Arrow economy with growth,* $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t, z_{t+1}) e_{t+1}(z^{t+1}) \text{ with } \hat{a}_t \text{ defined in } (36) \\
v_t(z_t) &= \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1}|z_t)}{\lambda(z_{t+1})} \left[\frac{v_{t+1}(z_{t+1}) + \alpha e_{t+1}(z_{t+1})}{\hat{R}_t^A(z_t)} \right] \\
\hat{R}_t^A(z_t) &= \frac{\hat{R}_t \widetilde{\beta}_{t+1}}{\hat{\beta}(z_t) \widetilde{\beta}_t} \\
q_t(z^t, z_{t+1}) &= \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{1}{\hat{R}_t^A(z_t)} * \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}
\end{aligned}$$

[31] Without this argument we merely would have shown that a Bewley equilibrium for initial condition Θ_0 can be implemented as equilibrium of the de-trended Arrow model with intial conditions z_0 and some initial distribution of wealth, but not necessarily Θ_0. For an equivalence result this seems insufficient.

Risk Premia Since we have shown that the equivalence results from section 4 survive the introduction of non-i.i.d. aggregate shocks (provided that a complete menu of aggregate-state-contingent securities can be traded and solvency constraints are not binding), this implies that our baseline results for risk premia survives as well.[32] We demonstrated that endowment shocks translate into aggregate taste shocks; but these only affect interest rates and price/dividend ratios, but not risk premia. When all agents become more impatient, the interest rises and the price/dividend ratio decreases, but the conditional expected excess return remains unchanged.

Solvency Constraints So far we have abstracted from binding solvency constraints. So far we had assumed solvency constraints of the form $K_t(s^t) = \hat{K}_t(y^t)e_t(z^t)$ or $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t)e_t(z^{t+1})$ and used the constraints $\hat{K}_t(y^t)$ and $\hat{M}_t(y^t)$ in the de-trended versions of the model. Since the Arrow securities positions in (36) now vary with z_t we cannot guarantee that the asset positions in the de-trended Arrow model implied by the Bewley equilibrium satisfy the solvency constraints in that model, and thus the allocations stated in theorem 7.2 satisfy the original constraints defined by $K_t(s^t)$ and $M_{t+1}(s^{t+1})$. However, we can still state

Proposition 7.2. *The allocations from theorem 7.2 satisfy the modified solvency constraints:*

$$\begin{aligned} K_t^*(s^t) &= K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(\theta_0, s^t, z_{t+1}) + \sigma_t(\theta_0, s^t) \left[v_t(z_t) - \hat{v}_t e_t(z^t) \right] && (37) \\ M_{t+1}^*(s^{t+1}) &= M_{t+1}(s^{t+1}) + a_t(\theta_0, s^t, z_{t+1}) + \sigma_t(\theta_0, s^t) \left[v_{t+1}(z_{t+1}) - \hat{v}_{t+1} e_{t+1}(z^{t+1}) \right] && (38) \end{aligned}$$

where $\hat{v}_t$ is the (deterministic) Bewley equilibrium stock price.

Consequently in the case of non-i.i.d. aggregate shocks our implementation and asset pricing results are not fully robust to the introduction of binding solvency constraints. The allocations derived from the Bewley model satisfy the modified solvency constraints (37) and (38) in the growing Arrow model, but not the original constraints we have specified.[33]

[32] Note that
$$\kappa_t(z_t) = \frac{\hat{R}_t^{RE}(z_t)}{\hat{R}_t^A(z_t)} = \frac{\widetilde{\beta}_t}{\hat{R}_t \widetilde{\beta}_{t+1}} = \kappa_t$$
is still deterministic, and thus the proofs of section 6 go through unchanged.

[33] These violations of the original constraints are completely due to the impact of interest rate shocks on the value of the asset portfolio.They ought to be small as long as interest rates do not vary too much over the business cycle, because then the Arrow securities positions needed to hedge against the interest rate shocks are small and the capital loss or gain of stock prices $v_t(z_t) - \hat{v}_t e_t(z^t)$ stemming from interest rate shocks are small.

7.1.3 *THL* Model

For the *THL* model our previous equivalence result no longer holds since with predictability in aggregate consumption growth households trade state-contingent claims in the equilibrium of the Arrow model. The market structure of the *THL* model prevents them from doing so, and thus our implementation and irrelevance results in this model are not robust to the introduction of non-i.i.d aggregate endowment growth.

7.2 Preferences

So far we assumed a time-separable lifetime utility function where the felicity function was of the CRRA variety. In section (A.1) of the appendix we show that our results go through with Epstein-Zin utility and i.i.d. aggregate endowment growth. This suggests that it is homotheticity, rather than time separability of the utility function that is crucial for our results.

8 Conclusion

Recently Krusell and Smith (1997) and Storesletten, Telmer and Yaron (2006) have argued that models with idiosyncratic income shocks and incomplete markets can generate an equity premium that is substantially larger than the CCAPM if there is counter-cyclical cross-sectional variance (CCV) in labor income shocks. Storesletten, Telmer and Yaron (2004) argue that this condition is satisfied in the data, although it is still open to debate whether CCV in the data is strong enough to explain equity risk premia at reasonable levels of risk aversion. The main result of this paper is to show analytically that for a general class of models CCV is not only sufficient, but *necessary* to make uninsurable idiosyncratic income shocks a potentially useful model element for explaining the equity premium.

Our results are complementary to the findings by Mankiw (1986) . He showed that if the marginal utility of consumption is convex (a property that CRRA as well as CARA utility satisfies) then larger risk premia than in the representative agent model are obtained, *if* the cross-sectional variance of equilibrium household consumption growth increases in recessions. Our results show that this cross-sectional variance of consumption is *constant over the cycle* if the distribution of idiosyncratic income shocks is independent of aggregate conditions. Thus our work shows that solvency constraints and transaction costs in incomplete market models alone, without the CCV mechanism cannot produce this time variation in the cross-sectional consumption variance as an equilibrium outcome. From the perspective of these results therefore future work ought to focus on investigating what deviations from our assumptions can give idiosyncratic income risk a prominent role to play in explaining the market price of risk.

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A Additional Definitions

A.1 Arrow Model

The definition of an equilibrium in the Arrow model is standard. Each household is assigned a label that consists of its initial financial wealth θ_0 and its initial state $s_0 = (y_0, z_0)$. A household of type (θ_0, s_0) then chooses consumption allocations $\left\{c_t(\theta_0, s^t)\right\}$, trading strategies for Arrow securities $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$ and shares $\{\sigma_t(\theta_0, s^t)\}$ to maximize her expected utility (1), subject to the budget constraints (11) and subject to solvency constraints (13) or (14).

Definition A.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the Arrow model consists of household allocations $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\left\{c_t(\theta_0, s^t)\right\}$ and prices $\left\{q_t(z^t, z_{t+1})\right\}$, $\{v_t(z^t)\}$ such that*

1. *Given prices, household allocations solve the household maximization problem*

2. *The goods market clears for all z^t,*

$$\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} c_t(\theta_0, s^t) d\Theta_0 = e_t(z^t)$$

3. *The asset markets clear for all z^t*

$$\begin{aligned}
\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\
\int \sum_{y^t} \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t | z_0)} a_t(\theta_0, s^t, z_{t+1}) d\Theta_0 &= 0 \text{ for all } z_{t+1} \in Z
\end{aligned}$$

A.1.1 Optimality Conditions for De-trended Arrow Model

Define Lagrange multiplier

$$\hat{\beta}(s^t) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \hat{\mu}(s^t) \geq 0$$

for the constraint in (16) and

$$\hat{\beta}(s^t) \hat{\pi}(s^t | s_0) u'(\hat{c}_t(s^t)) \hat{\kappa}_t(s^t, z_{t+1}) \geq 0$$

for the constraint in (17). The Euler equations of the de-trended Arrow model are given by:

$$\begin{aligned}
1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1} | s^t, z_{t+1}} \hat{\pi}(s_{t+1} | s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&\quad + \hat{\mu}_t(s^t) + \frac{\hat{\kappa}_t(s^t, z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \ \forall z_{t+1}. \qquad (39) \\
1 &= \hat{\beta}(s_t) \sum_{s^{t+1} | s^t} \hat{\pi}(s_{t+1} | s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&\quad + \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right]. \qquad (40)
\end{aligned}$$

Only one of the two Lagrange multipliers enters the equations, depending on which version of the solvency constraint we consider. The complementary slackness conditions for the Lagrange multipliers are given by

$$\begin{aligned}
\hat{\mu}_t(s^t)\left[\sum_{z_{t+1}}\hat{a}_t(s^t,z_{t+1})\hat{q}_t(z^t,z_{t+1})+\hat{\sigma}_t(s^t)\hat{v}_t(z^t)-\hat{K}_t(y^t)\right] &= 0\\
\hat{\kappa}_t(s^t,z_{t+1})\left[\hat{a}_t(s^t,z_{t+1})+\hat{\sigma}_t(s^t)\left[\hat{v}_{t+1}(z^{t+1})+\alpha\right]-\hat{M}_t(y^t)\right] &= 0.
\end{aligned}$$

The appropriate transversality conditions read as

$$\begin{aligned}
\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))[\hat{a}_{t-1}(s^{t-1},z_t)-\hat{M}_{t-1}(y^{t-1})] &= 0\\
\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))[\hat{\sigma}_{t-1}(s^{t-1})(\hat{v}_t(z^t)+\alpha)-\hat{M}_{t-1}(y^{t-1})] &= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\sum_{z_{t+1}}\hat{a}_t(s^t,z_{t+1})\hat{q}_t(z^t,z_{t+1})-\hat{K}_t(y^t)\right] &= 0\\
\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\hat{\sigma}_t(s^t)\hat{v}_t(z^t)-\hat{K}_t(y^t)\right] &= 0.
\end{aligned}$$

Since the household optimization has a concave objective function and a convex constraint set the first order conditions and complementary slackness conditions, together with the transversality condition, are necessary and sufficient conditions for optimality of household allocation choices.

A.2 *THL* Model

Agents only trade a single bond a single stock. Wealth tomorrow in state $s^{t+1}=(s^t,y_{t+1},z_{t+1})$ is given by

$$\theta_{t+1}(s^{t+1})=\eta(y_{t+1})e_{t+1}(z_{t+1})+b_t(s^t)+\sigma_t(s^t)\left[v_{t+1}(z^{t+1})+\alpha e_{t+1}(z_{t+1})\right].$$

Definition A.2. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0,y_0), a competitive equilibrium for the* THL *economy consists of household allocations $\{b_t(\theta_0,s^t)\}$, $\{c_t(\theta_0,s^t)\}$, $\{\sigma_t(\theta_0,s^t)\}$ and interest rates $\{R_t(z^t)\}$ and share prices $\{v_t(z^t)\}$ such that*

1. *Given prices, allocations solve the household maximization problem*
2. *The goods market clears for all z^t*

$$\int\sum_{y^t}\frac{\pi(y^t,z^t|y_0,z_0)}{\pi(z^t|z_0)}c_t(\theta_0,s^t)d\Theta_0=e_t(z^t)$$

3. The asset markets clear for all z^t

$$\begin{aligned}
\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\
\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} b_t(\theta_0, s^t) d\Theta_0 &= 0.
\end{aligned}$$

A.2.1 Optimality Conditions for *THL* Economy

Define the Lagrange multiplier

$$\hat{\beta}(s^t)\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\hat{\mu}(s^t) \geq 0$$

for the constraint in (28) and

$$\hat{\beta}(s^t)\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\hat{\kappa}_t(s^t, z_{t+1}) \geq 0$$

for the constraint in (29). In the detrended *THL* economy the Euler equations read as

$$\begin{aligned}
1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&\quad + \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \qquad (41) \\
1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&\quad + \hat{\mu}_t(s^t) + \sum_{z_{t+1}} \hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right], \qquad (42)
\end{aligned}$$

with complementary slackness conditions given by:

$$\begin{aligned}
\hat{\mu}_t(s^t) \left[\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \hat{\sigma}_t(s^t)\hat{v}_t(z^t) - \hat{K}_t(y^t)\right] &= 0 \\
\hat{\kappa}_t(s^t, z_{t+1}) \left[\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \hat{\sigma}_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] - \hat{M}_t(y^t)\right] &= 0
\end{aligned}$$

The transversality conditions read as

$$\begin{aligned}
\lim_{t\to\infty} \sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t)) \left[\frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} - \hat{M}_{t-1}(y^{t-1})\right] &= 0 \\
\lim_{t\to\infty} \sum_{s^t} \hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))[\hat{\sigma}_{t-1}(s^{t-1})(\hat{v}_t(z^t) + \alpha) - \hat{M}_{t-1}(y^{t-1})] &= 0
\end{aligned}$$

and

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\frac{\hat{b}_t(s^t)}{R_t(z^t)}-\hat{K}_t(y^t)\right] = 0$$
$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}(s^{t-1})\hat{\pi}(s^t|s_0)u'(\hat{c}_t(s^t))\left[\hat{\sigma}_t(s^t)\hat{v}_t(z^t)-\hat{K}_t(y^t)\right] = 0.$$

A.3 Representative Agent Model

The budget constraint of the representative agent who consumes aggregate consumption $c_t(z^t)$ reads as

$$\begin{aligned} & c_t(z^t)+\sum_{z_{t+1}}a_t(z^t,z_{t+1})q_t(z^t,z_{t+1})+\sigma_t(z^t)v_t(z^t) \\ \leq\ & e_t(z^t)+a_{t-1}(z^{t-1},z_t)+\sigma_{t-1}(z^{t-1})\left[v_t(z^t)+\alpha e_t(z_t)\right] \end{aligned}$$

After deflating by the aggregate endowment $e_t(z^t)$, the budget constraint reads as

$$\begin{aligned} & \hat{c}_t(z^t)+\sum_{z_{t+1}}\hat{a}_t(z^t,z_{t+1})\hat{q}_t(z^t,z_{t+1})+\sigma_t(z^t)\hat{v}_t(z^t) \\ \leq\ & 1+\hat{a}_{t-1}(z^{t-1},z_t)+\sigma_{t-1}(z^{t-1})\left[\hat{v}_t(z^t)+\alpha\right], \end{aligned}$$

where $\hat{a}_t(z^t,z_{t+1})=\frac{a_t(z^t,z_{t+1})}{e_{t+1}(z^{t+1})}$ and $\hat{q}_t(z^t,z_{t+1})=q_t(z^t,z_{t+1})\lambda(z_{t+1})$ as well as $\hat{v}_t(z^t)=\frac{v_t(z^t)}{e_t(z^t)}$, precisely as in the Arrow model. Obviously, in an equilibrium of this model the representative agent consumes the aggregate endowment.

Lemma A.1. *Equilibrium asset prices are given by*

$$\begin{aligned} \hat{q}_t(z^t,z_{t+1}) &= \hat{\beta}\hat{\phi}(z_{t+1})=\hat{q}(z_{t+1}) \text{ for all } z_{t+1} \\ \hat{v}_t(z^t) &= \hat{\beta}\sum_{z_{t+1}}\hat{\phi}(z_{t+1})\left[\hat{v}_{t+1}(z^{t+1})+\alpha\right] \end{aligned}$$

A.4 Recursive Utility

We consider the class of preferences due to Epstein and Zin (1989) . Let $V(c^i)$ denote the utility derived from consuming c^i :

$$V(c^i)=\left[(1-\beta)c_t^{1-\rho}+\beta(\mathcal{R}_tV_1)^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where the risk-adjusted expectation operator is defined as:

$$\mathcal{R}_tV_{t+1}=\left(E_tV_{t+1}^{1-\alpha}\right)^{1/1-\alpha}.$$

α governs risk aversion and ρ governs the willingness to substitute consumption intertemporally. These preferences impute a concern for the timing of the resolution of uncertainty to agents. In the special case where $\rho=\frac{1}{\alpha}$, these preferences collapse to standard power utility preferences with CRRA coefficient α. As before, we can define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}$$

$$\text{and } \hat{\beta}(s_t) = \beta \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}}$$

As before, $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. Note that the adjustment of the discount rate is affected by both ρ and α. If $\rho = \frac{1}{\alpha}$, this transformation reduces to the case we discussed in section (2).

Finally, let $\hat{V}_t(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node s^t, under the new transition probabilities and discount factor, defined over consumption shares $\left\{\hat{c}_t(s^t)\right\}$:

$$\hat{V}_t(\hat{c})(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t \hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where $\mathcal{R}$ denotes the following operator:

$$\hat{\mathcal{R}}_t V_{t+1} = \left(\hat{E}_t \hat{V}_{t+1}^{1-\alpha}\right)^{1/1-\alpha}.$$

and $\hat{E}$ denotes the expectation operator under the hatted measure $\hat{\pi}$.

Proposition A.1. *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any s^t and any two consumption allocations c, c'*

$$V(c)(s^t) \geq V(c')(s^t) \Longleftrightarrow \hat{V}(\hat{c})(s^t) \geq \hat{V}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (4).

Detrended Arrow Economy We proceed as before, by conjecturing that the equilibrium consumption shares only depend on y^t. Our first result states that if the consumption shares in the de-trended economy do not depend on the aggregate history z^t, then it follows that the interest rates in this economy are deterministic.

Proposition A.2. *In the de-trended Arrow economy, if there exists a competitive equilibrium with equilibrium consumption allocations $\left\{\hat{c}_t(\theta_0, y^t)\right\}$, then there is a deterministic interest rate process $\left\{\hat{R}_t^A\right\}$ and equilibrium prices $\left\{\hat{q}_t(z^t, z_{t+1})\right\}$, that satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \tag{43}$$

All the results basically go through. We can map an equilibrium of the Bewley economy into an equilibrium of the detrended Arrow economy.

Theorem A.1. *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ can be made into an equilibrium for the Arrow economy with growth, $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$, $\{\sigma_t(\theta_0, s^t)\}$,*

$\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t)e_t(z^t) \\ \sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\ a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t)e_{t+1}(z^{t+1}) \\ v_t(z^t) &= \hat{v}_t e_t(z^t) \\ q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\alpha}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\alpha}} \end{aligned}$$

As a result, even for an economy with agents who have these Epstein-Zin preferences, the risk premium is not affected.

B Proofs

- Proof of Proposition 2.1:

Proof. Denote $U(c)(s^t)$ as continuation utility of an agent from consumption stream c, starting at history s^t. This continuation utility follows the simple recursion

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)U(c)(s^t, s_{t+1}),$$

where it is understood that $(s^t, s_{t+1}) = (z^t, z_{t+1}, y^t, y_{t+1})$. Divide both sides by $e_t(s^t)^{1-\gamma}$ to obtain

$$\frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\frac{e_{t+1}(z^{t+1})^{1-\gamma}}{e_t(z^t)^{1-\gamma}}\frac{U(c)(s^t, s_{t+1})}{e_{t+1}(z^{t+1})^{1-\gamma}}.$$

Define a new continuation utility index $\hat{U}(\cdot)$ as follows:

$$\hat{U}(\hat{c})(s^t) = \frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}}.$$

It follows that

$$\begin{aligned} \hat{U}(\hat{c})(s^t) &= u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}\hat{U}(\hat{c})(s^t, s_{t+1}) \\ &= u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t)\hat{U}(\hat{c})(s^t, s_{t+1}) \end{aligned}$$

Thus it follows, for two consumption streams c and c', that

$$U(c)(s^t) \geq U(c')(s^t) \text{ if and only if } \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

i.e. the household orders original and growth-deflated consumption streams in exactly the same way. □

- Proof of Theorem 4.1:

Proof. The proof consists of two parts. In a first step, we argue that Bewley equilibrium allocations and prices can be made into an equilibrium for the de-trended Arrow model, and in a second step, we argue that by scaling the allocations and prices by the appropriate endowment (growth) factors results in an equilibrium of the stochastically growing Arrow model.

Step 1: Take allocations and prices from a Bewley equilibrium, $\left\{\hat{c}_t(y^t), \hat{a}_t(y^t), \hat{\sigma}_t(y^t)\right\}, \{\hat{R}_t, \hat{v}_t\}$ and let the associated Lagrange multipliers on the solvency constraints be given by

$$\hat{\beta}^t \varphi(y^t|y_0) u'(\hat{c}_t(y^t))\hat{\mu}(y^t) \geq 0$$

for the constraint in (9) and

$$\hat{\beta}^t \varphi(y^t|y_0) u'(\hat{c}_t(y^t))\hat{\kappa}_t(y^t) \geq 0$$

for the constraint in (10). The first order conditions (which are necessary and sufficient for household optimal choices together with the complementary slackness and transversality conditions) in the Bewley model, once combined to the Euler equations, read as

$$\begin{aligned} 1 &= \hat{R}_t \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t \hat{\kappa}_t(y^t) && (44) \\ &= \hat{\beta} \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}\right] \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}\right] \hat{\kappa}_t(y^t). && (45) \end{aligned}$$

The corresponding Euler equations for the de-trended Arrow model, evaluated at the Bewley equilibrium allocations and Lagrange multipliers $\hat{\mu}(y^t)$ and $\hat{\kappa}_t(y^t)\hat{\phi}(z_{t+1})$, read as (see (39) and (40)):

$$\begin{aligned} 1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s^{t+1}|s^t, z_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \hat{\mu}_t(y^t) + \frac{\hat{\kappa}_t(y^t)\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \ \forall z_{t+1}. \\ 1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right]. \end{aligned}$$

Evaluated at the conjectured prices

$$\begin{aligned} \hat{v}_t(z^t) &= \hat{v}_t && (46) \\ \hat{q}_t(z^t, z_{t+1}) &= \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t}, && (47) \end{aligned}$$

and using the independence and i.i.d. assumptions, which imply (i)

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1})$$

and (ii) $\hat{\beta}(s_t) = \hat{\beta}$ these Euler equations can be restated as follows:

$$
\begin{aligned}
1 &= \frac{\hat{\beta}\hat{R}_t}{\hat{\phi}(z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}_t(y^t) + \hat{R}_t\hat{\kappa}_t(y^t) \\
1 &= \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\
&+ \hat{\mu}_t(y^t) + \hat{\kappa}_t(y^t) \left[\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}\right] \sum_{z_{t+1}} \hat{\phi}(z_{t+1}),
\end{aligned}
$$

which are, given that $\sum_{z_{t+1}} \hat{\phi}(z_{t+1}) = 1$, exactly the Euler conditions (44) and (45) of the Bewley model and hence satisfied by the Bewley equilibrium allocations. A similar argument applies to the complementary slackness conditions, which for the Bewley model read as

$$
\hat{\mu}_t(y^t) \left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t - \hat{K}_t(y^t)\right] = 0 \tag{48}
$$

$$
\hat{\kappa}_t(y^t) \left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) - \hat{M}_t(y^t)\right] = 0 \tag{49}
$$

and for the de-trended Arrow model, evaluated at Bewley equilibrium allocations and conjectured prices, read as

$$
\begin{aligned}
\hat{\mu}_t(y^t) \left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t) - \hat{K}_t(y^t)\right] &= 0 \\
\hat{\kappa}_t(y^t) \left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] - \hat{M}_t(y^t)\right] &= 0/\hat{\phi}(z_{t+1})
\end{aligned}
$$

Again, the Bewley equilibrium allocations satisfy the complementary slackness conditions in the de-trended Arrow model. The argument is exactly identical for the transversality conditions. Finally, we have to check whether the Bewley equilibrium allocation satisfies the de-trended Arrow budget constraints. Plugging in the allocations yields

$$
\hat{c}_t(s^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) + \hat{\sigma}_t(y^t)\hat{v}_t \leq \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right], \tag{50}
$$

which is exactly the budget constraint in the Bewley model. Thus, given the conjectured prices Bewley equilibrium allocations are optimal household choices also in the de-trended Arrow model.

Since the market clearing conditions for assets and consumption goods coincide in the two models, Bewley allocations satisfy the market clearing conditions in the de-trended Arrow model. Thus we conclude that Bewley equilibrium allocations, together with prices (46) and (47) are an equilibrium in the de-trended Arrow model.

Step 2: Now we need to show that an equilibrium of the de-trended Arrow model is, after appropriate scaling, an equilibrium in the stochastically growing economy, but this was established in section 4.2 where we showed that by with the transformations $\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)}$, $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$, $\hat{\sigma}_t(s^t) = \sigma_t(s^t)$, $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$, $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$

household problems and market clearing conditions in the de-trended and the stochastically growing Arrow model coincide. ☐

- Proof of Theorem 5.1:

Proof. As in the Arrow economy, the crucial part of the proof is to argue that Bewley equilibrium allocations and prices can be made into an equilibrium for the de-trended *THL* model. The Euler equations of the Bewley model where given in (44) and (45).

The corresponding Euler equations for the de-trended *THL* model, evaluated at the Bewley equilibrium allocations and Lagrange multipliers $\hat{\mu}(y^t)$ and $\hat{\kappa}_t(y^t)\hat{\phi}(z_{t+1})$, read as (see (41) and (42))

$$\begin{aligned}
1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\
&\quad + \hat{\mu}(y^t) + \hat{\kappa}_t(y^t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \\
1 &= \hat{\beta}(s_t) \sum_{s^{t+1}|s^t} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\
&\quad + \hat{\mu}(y^t) + \hat{\kappa}_t(y^t) \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right].
\end{aligned}$$

Evaluated at the conjectured prices, we obtain:

$$\begin{aligned}
\hat{v}_t(z^t) &= \hat{v}_t \\
R_t(z^t) &= \hat{R}_t * \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}
\end{aligned}$$

and using the independence and i.i.d. assumptions, we obtain (i) $\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1})$ and (ii) $\hat{\beta}(s_t) = \hat{\beta}$ and, by definition of $\hat{\phi}(z_{t+1})$,

$$R_t(z^t) \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} = R_t(z^t) \frac{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}} = \hat{R}_t$$

these equations can be restated as:

$$\begin{aligned}
1 &= \hat{\beta}\hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} + \hat{\mu}(y^t) + \hat{\kappa}_t(y^t)\hat{R}_t \\
1 &= \hat{\beta} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t) \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}\right] \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \\
&\quad + \hat{\mu}(y^t) + \hat{\kappa}_t(y^t) \left[\frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}\right] \sum_{z_{t+1}} \hat{\phi}(z_{t+1}),
\end{aligned}$$

which again are, given that $\sum_{z_{t+1}} \hat{\phi}(z_{t+1}) = 1$, exactly the Euler conditions (44) and (45) of the Bewley model and hence satisfied by the Bewley equilibrium allocations. For the Bewley model, the complementary slackness conditions were given in (48) and (49), and for the de-trended *THL* model, evaluated at the proposed allocations in the theorem (which had bond holdings equal to zero) these equations are given by:

$$\begin{aligned}
\hat{\mu}_t(y^t)\left[\hat{\sigma}_t^{THL}(y^t)\hat{v}_t - \hat{K}_t(y^t)\right] &= \hat{\mu}_t(y^t)\left[\left(\frac{\hat{a}_t(y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(y^t)\right)\hat{v}_t - \hat{K}_t(y^t)\right] \\
&= \hat{\mu}_t(y^t)\left[\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t - \hat{K}_t(y^t)\right] = 0
\end{aligned}$$

and

$$\begin{aligned}
\hat{\kappa}_t(y^t)\left[\hat{\sigma}_t^{THL}(y^t)\left[\hat{v}_{t+1}+\alpha\right] - \hat{M}_t(y^t)\right] &= \hat{\kappa}_t(y^t)\left[\left(\frac{\hat{a}_t(y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(y^t)\right)\left[\hat{v}_{t+1}+\alpha\right] - \hat{M}_t(y^t)\right] \\
&= \hat{\kappa}_t(y^t)\left[\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1}+\alpha\right] - \hat{M}_t(y^t)\right] = 0/\hat{\phi}(z_{t+1})
\end{aligned}$$

where we use the fact that the Bewley equilibrium prices and interest rates satisfy

$$\hat{R}_t = \frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t}$$

These complementary slackness conditions are satisfied since the Bewley equilibrium allocations satisfy the complementary slackness conditions in the Bewley model. The argument is exactly identical for the transversality conditions. Finally we have to check whether the allocations proposed in the theorem satisfy the de-trended *THL* model budget constraints. Plugging into the de-trended *THL* model budget constraint yields

$$\begin{aligned}
\hat{c}_t(y^t) + \frac{\hat{b}_t(y^t)}{R_t} + \hat{\sigma}_t^{THL}(y^t)\hat{v}_t &\leq \eta(y_t) + \frac{\hat{b}_{t-1}(y^{t-1})}{\lambda(z_t)} + \hat{\sigma}_{t-1}^{THL}(y^{t-1})\left[\hat{v}_t+\alpha\right] \qquad (51) \\
\hat{c}_t(y^t) + \left[\frac{\hat{a}_t(y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(y^t)\right]\hat{v}_t &\leq \eta(y_t) + \left[\frac{\hat{a}_{t-1}(y^{t-1})}{[\hat{v}_t+\alpha]} + \hat{\sigma}_{t-1}(y^{t-1})\right]\left[\hat{v}_t+\alpha\right] \\
\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t &\leq \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})\left[\hat{v}_t+\alpha\right]
\end{aligned}$$

which is exactly the budget constraint in the Bewley model. Thus, given the conjectured prices the allocations proposed in the theorem are optimal household choices in the de-trended *THL* model. Note that equation (51) shows why, in contrast to the Arrow model, in the *THL* model bond positions have to be zero. Nothing in this equation depends in the aggregate shock z_t but the term $\frac{\hat{b}_{t-1}(y^{t-1})}{\lambda(z_t)}$. Therefore the budget constraint can only be satisfied if $\hat{b}_{t-1}(y^{t-1}) = 0$.[34]

The market clearing conditions for bonds in the de-trended *THL* model is trivially satisfied because bond positions are identically equal to zero. The goods market clearing condition is identical to that of the Bewley model and thus satisfied by the Bewley equilibrium consump-

[34]Intuitively, in the model with growth households want to keep wealth at the beginning of the period proportional to the aggregate endowment in the economy. But since bond positions are chosen yesterday and thus cannot depend on the realization of the aggregate shock today bond positions have to be zero to achieve proportionally of wealth and the aggregate endowment.

tion allocations. It remains to be shown that the stock market clears. But

$$\begin{aligned}
& \int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t^{THL}(\theta_0, y^t) d\Theta_0 \\
= & \int \sum_{y^t} \varphi(y^t|y_0) \left[\frac{\hat{a}_t(\theta_0, y^t)}{[\hat{v}_{t+1}+\alpha]} + \hat{\sigma}_t(\theta_0, y^t) \right] d\Theta_0 \\
= & \frac{1}{[\hat{v}_{t+1}+\alpha]} \int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t) d\Theta_0 + \int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t) d\Theta_0 \\
= & 0 + 1
\end{aligned}$$

where the last line follows from the fact that the bond and stock market clears in the Bewley equilibrium. Thus we conclude that the allocations and prices proposed in the theorem form indeed an equilibrium in the de-trended *THL* model, and, after appropriate scaling, in the original *THL* model. □

Proof of Lemma 6.1:

Proof. The stock return is defined as:

$$R_{t+1}^s(z^{t+1}) = \frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})}{v_t(z^t)}.$$

Subtracting the two Euler equations (39)-(40) in the Arrow model and (41)-(42) in the *THL* model yields, in both cases

$$\hat{\beta} \sum_{z^{t+1}|z^t} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} - \hat{R}_t(z^t) \right] = 0.$$

Using the fact that $\hat{v}_{t+1}(z^{t+1}) = v_{t+1}(z^{t+1})/e_{t+1}(z_{t+1})$ and the definition of $\hat{\phi}(z_{t+1})$ and $\hat{\beta}$, as well as (23) yields

$$\beta \sum_{z^{t+1}|z^t} \phi(z_{t+1})\lambda(z_{t+1})^{-\gamma} \left[R_{t+1}^s(z^{t+1}) - R_t(z^t) \right] = 0$$

or in short

$$E_t \left\{ \beta\lambda(z_{t+1})^{-\gamma} \left[R_{t+1}^s - R_t \right] \right\} = 0$$

Thus the representative agent stochastic discount factor $\beta\lambda(z_{t+1})^{-\gamma}$ prices the excess return of stocks over bonds in both the Arrow and the *THL* model. Note that in the Arrow model (but not in the *THL* model) this stochastic discount factor any excess return $R_{t+1}^i - R_t$ as long as the returns only depend on the aggregate state z_{t+1}.

□

- Proof of Proposition 6.1:

Proof. From theorem 4.1 we know that in the Arrow model equilibrium prices for Arrow securities are given by:

$$q_t^A(z^t, z_{t+1}) = \frac{\hat{q}_t^A(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})\hat{R}_t^A}$$

whereas in the representative agent model equilibrium prices for Arrow securities are given by:

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \hat{\beta}\frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})}$$

so that

$$\frac{q_t^A(z^t, z_{t+1})}{q_t(z^t, z_{t+1})} = \frac{1}{\hat{\beta}\hat{R}_t^A} = \frac{\hat{R}^{RE}}{\hat{R}_t^A} = \kappa_t \geq 1 \tag{52}$$

where

$$\hat{R}_t^{RE} = \hat{R}^{RE} = \frac{1}{\sum\limits_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})} = \frac{1}{\sum\limits_{z_{t+1}} \hat{\beta}\hat{\phi}(z_{t+1})} = \frac{1}{\hat{\beta}}$$

is the risk-free interest rate in the de-trended representative agent model. Note that the multiplicative factor κ_t may depend on time since $\hat{R}_t^A$ may, but is nonstochastic, since $\hat{R}_t^A = \hat{R}_t$ (the risk-free interest rate in the de-trended Arrow model equals that in the Bewley model, which is evidently nonstochastic). Since interest rates in the Bewley model are (weakly) smaller than in the representative agent model, $\kappa_t \geq 1$. Equation (52) implies that the stochastic discount factor in the Arrow model equals the SDF in the representative agent model, multiplied by κ_t:

$$m_{t+1}^A(z^{t+1}) = m_{t+1}^{RE}(z^{t+1})\kappa_t$$

Finally, since the stochastic discount factor for the Arrow model is also a valid stochastic discount factor in the *THL* model (although not necessarily the unique valid stochastic discount factor), the previous result also applies to the *THL* model. □

- Proof of Theorem 6.1:

Proof. Remember that we defined the multiplicative risk premium in the main text as

$$1 + \nu_t = \frac{E_t R_{t,1}\left[\{e_{t+k}\}\right]}{R_{t,1}\left[1\right]}$$

We use $m_{t,t+k} = m_{t+1} \cdot m_{t+2} \ldots \cdot m_{t+k}$ to denote the k-period ahead pricing kernel (with convention that $m_{t,t} = 1$), such that $E_t(d_{t+k}m_{t,t+k})$ denotes the price at time t of a random payoff d_{t+k}. Note that whenever there is no room fro confusion we suppress the dependence of variables on z^t.

First, note that the multiplicative risk premium on a claim to aggregate consumption can be stated as a weighted sum of risk premia on strips (as shown by Alvarez and Jermann (2001)).

By definition of $R_{t,1}\left[\{e_{t+k}\}\right]$ we have

$$
\begin{aligned}
R_{t,1}\left[\{e_{t+k}\}\right] &= \frac{\sum_{k=1}^{\infty} E_{t+1} m_{t+1,t+k} \alpha e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} \alpha e_{t+k}} && (53)\\
&= \frac{1}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}} * \sum_{k=1}^{\infty} \frac{E_{t+1} m_{t+1,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}} * E_t m_{t,t+k} e_{t+k} && (54)\\
&= \sum_{k=1}^{\infty} \frac{E_{t+1} m_{t+1,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}} \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}} \\
&= \sum_{k=1}^{\infty} \omega_k R_{t,1}\left[e_{t+k}\right] && (55)
\end{aligned}
$$

where the nonrandom weights ω_k are given by

$$
\omega_k = \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{j=1}^{\infty} E_t m_{t,t+j} e_{t+j}}
$$

Thus

$$
1 + \nu_t = \frac{E_t R_{t,1}\left[\{e_{t+k}\}\right]}{R_{t,1}\left[1\right]} = \sum_{k=1}^{\infty} \omega_k \frac{E_t R_{t,1}\left[e_{t+k}\right]}{R_{t,1}\left[1\right]} \tag{56}
$$

and it is sufficient to show that the multiplicative risk premium $E_t R_{t,1}\left[e_{t+k}\right] / R_{t,1}\left[1\right]$ on all k-period strips of aggregate consumption (a claim to the Lucas tree's dividend in period k only, not the entire stream) is the same in the Arrow model as in the representative agent model. First, we show that the one-period ahead conditional strip risk premia are identical:

$$
\frac{E_t \frac{E_{t+1} e_{t+1}}{E_t[m^A_{t+1} e_{t+1}]}}{\frac{1}{E_t[m^A_{t+1}]}} = \frac{E_t \frac{E_{t+1} \lambda_{t+1}}{E_t[m^A_{t+1} \lambda_{t+1}]}}{\frac{1}{E_t[m^A_{t+1}]}} = \frac{E_t \frac{E_{t+1} \lambda_{t+1}}{E_t[m^{RE}_{t+1} \lambda_{t+1}]}}{\frac{1}{E_t[m^{RE}_{t+1}]}}
$$

The first equality follows from dividing through by e_t. The second equality follows from the expression for m^A in Proposition 6.1: $m^A_{t+1} = m^{RE}_{t+1} \kappa_t$.

Next we repeat the argument for the risk premium of a k-period strip:

$$
\frac{E_t R^A_{t,1}\left[e_{t+k}\right]}{E_t R^A_{t,1}\left[1\right]} = \frac{E_t \frac{E_{t+1}[m^A_{t+1,t+k} e_{t+k}]}{E_t[m^A_{t,t+k} e_{t+k}]}}{\frac{1}{E_t[m^A_{t+1}]}} = \frac{E_t \frac{\kappa_{t+1} \cdot \ldots \cdot \kappa_{t+k-1}}{\kappa_t \cdot \kappa_{t+1} \cdot \ldots \cdot \kappa_{t+k-1}} \frac{E_{t+1}[m^{RE}_{t+1,t+k} e_{t+k}]}{E_t[m^{RE}_{t,t+k} e_{t+k}]}}{\frac{1}{\kappa_t E_t[m^{RE}_{t+1}]}} = \frac{E_t R^{RE}_{t,1}\left[e_{t+k}\right]}{R^{RE}_{t,1}\left[1\right]}
$$

and thus risk premia on all k-period consumption strips in the Arrow model coincide with those in the representative agent model. But then (56) implies that the multiplicative risk premium in the two models coincide as well. □

- Proof of Lemma 7.1:

Proof. Absent binding solvency constraints the Euler equation in the Bewley model read as

$$\begin{aligned} 1 &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t}\hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t}\frac{\hat{v}_{t+1}+\alpha}{\hat{v}_t} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))} \end{aligned}$$

while in the Arrow model the Euler equations for Arrow securities are given by

$$1 = \frac{\hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)}{\hat{q}_t(z^t,z_{t+1})} \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

With conjectured Arrow securities prices $\hat{q}_t(z^t,z_{t+1}) = \hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)\frac{1}{\hat{R}_t}\frac{\widetilde{\beta}_t}{\widetilde{\beta}_{t+1}}$ these equations obviously coincide with the Bond Euler equation in the Bewley model, and thus the Bewley equilibrium allocation satisfies the Euler equations for Arrow securities. A similar argument applies to the Euler equation for stocks:

$$\begin{aligned} 1 &= \hat{\beta}(z_t) \sum_{z^{t+1}|z^t} \hat{\phi}(z_{t+1}|z_t)\left[\frac{\hat{v}_{t+1}(z^{t+1})+\alpha}{\hat{v}_t(z^t)}\right] \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &= \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t}\hat{R}_t \sum_{y^{t+1}|y^t} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t,y_{t+1}))}{u'(\hat{c}_t(y^t))} \end{aligned}$$

The state-contingent interest rate in this economy is given by:

$$\frac{1}{\hat{R}^A_t(z_t)} = \hat{\beta}(z_t)\frac{\widetilde{\beta}_t}{\hat{R}_t\widetilde{\beta}_{t+1}}$$

which can easily be verified from equation (33). □

- Proof of Proposition 7.1:

Proof. We need to check that Arrow securities positions defined in (36) satisfy the market clearing condition

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0)\hat{a}_{t-1}(\theta_0,y^{t-1},z^t)d\Theta_0 = 0 \text{ for all } z^t.$$

for each z_t. By the goods market clearing condition in the Bewley model we have, since total labor income makes up a fraction $1-\alpha$ of total income

$$
\begin{aligned}
& \int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \left(\hat{c}_t(y^t,\theta_0) - \eta(y_t)\right) d\Theta_0 \\
= & \int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \sum_{y^t|y^{t-1}} \varphi(y^t|y^{t-1}) \left(\hat{c}_t(y^t,\theta_0) - \eta(y_t)\right) d\Theta_0 \\
= & \sum_{y^t} \varphi(y^t|y_0) \left(\hat{c}_t(y^t,\theta_0) - \eta(y_t)\right) d\Theta_0 = \alpha
\end{aligned}
$$

Similarly

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \sum_{y^\tau} \varphi(y^\tau|y^t) \left[\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right] d\Theta_0 = \alpha \text{ for all } \tau > t$$

Since the stock market clears in the Bewley model we have

$$\int \sum_{y^{t-1}} \varphi(y^{t-1}|y_0) \hat{\sigma}_{t-1}(\theta_0, y^{t-1}) d\Theta_0 = 1.$$

Since the stock is a claim to α times the aggregate endowment in all future periods its (ex-dividend) price has to satisfy

$$\hat{v}_t(z_t) = \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t)$$

Combining these results implies that

$$
\begin{aligned}
& \int \sum_{y^t} \varphi(y^t|y_0) \hat{a}_{t-1}(\theta_0, y^{t-1}, z^t) d\Theta_0 \\
= \quad & \alpha + \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t) - 1 \left(\alpha + \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_t)\right) = 0
\end{aligned}
$$

for each z_t. Thus, each of the Arrow securities markets clears if households hold portfolios given by (36). $\square$

- Proof of Lemma 7.2:

Proof. By (32)

$$\hat{Q}_t(z^t|z_0) = \hat{\phi}(z^t|z_0) \tilde{Q}_t \frac{\hat{\beta}_{0,t-1}(z^{t-1}|z_0)}{\widetilde{\beta}_t}$$

and thus

$$
\begin{aligned}
\sum_{z^t} \hat{Q}_t(z^t|z_0) &= \frac{\tilde{Q}_t}{\widetilde{\beta}_t} \sum_{z^t|z_0} \hat{\phi}(z^t|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0) \\
&= \frac{\tilde{Q}_t}{\widetilde{\beta}_t} \sum_{z^t|z^{t-1}} \hat{\phi}(z^t|z^{t-1}) \sum_{z^{t-1}|z_0} \hat{\phi}(z^{t-1}|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0) = \tilde{Q}_t.
\end{aligned}
$$

by definition of $\widetilde{\beta}_t$ in equation (30) and the fact that $\sum_{z^t|z^{t-1}} \hat{\phi}(z^t|z^{t-1}) = 1$. □

Proof of Corollary 7.3:

Proof. The Arrow securities position at time zero needed to finance all future excess consumption mandated by the Bewley equilibrium is given by

$$
\begin{aligned}
\hat{a}_{-1}(\theta_0, y_0, z_0) &= \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_\tau(z^\tau|z_0) \left(\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right) \\
&\quad - \hat{\sigma}_0(\theta_0, y_0) \left[\hat{v}_0(z_0) + \alpha\right]
\end{aligned}
$$

where we substituted indexes $_{-1}$ by 0 to denote initial conditions. In particular, $\hat{\sigma}_0(\theta_0, y_0)$ is the initial share position of an individual with wealth θ_0. But

$$
\begin{aligned}
& \hat{a}_{-1}(\theta_0, y_0, z_0) \\
= \; & \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{z^\tau, y^\tau|z_0, y_0} \hat{Q}_\tau(z^\tau|z_0)\varphi(y^\tau|y_0) \left(\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right) - \hat{\sigma}_0(\theta_0, y_0) \left[\hat{v}_0(z_0) + \alpha\right] \\
= \; & \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \sum_{y^\tau|y_0} \varphi(y^\tau|y_0) \left(\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right) \sum_{z^\tau|z_0} \hat{Q}_\tau(z^\tau|z_0) \\
& - \hat{\sigma}_0(\theta_0, y_0)\alpha \left[1 + \sum_{\tau=1}^{\infty} \sum_{z^\tau|z_0} \hat{Q}_\tau(z^\tau|z_0)\right] \\
= \; & \hat{c}_0(\theta_0, y_0) - \eta(y_0) + \sum_{\tau=1}^{\infty} \widetilde{Q}_\tau \sum_{y^\tau|y_0} \varphi(y^\tau|y_0) \left(\hat{c}_\tau(\theta_0, y^\tau) - \eta(y_\tau)\right) - \hat{\sigma}_0(\theta_0, y_0)\alpha \left[\sum_{\tau=0}^{\infty} \widetilde{Q}_\tau\right] \\
= \; & 0
\end{aligned}
$$

where the last equality comes from the intertemporal budget constraint in the standard incomplete markets Bewley model and the fact that the initial share position in that model is given by $\hat{\sigma}_0(\theta_0, y_0)$. □

- Proof of of Lemma 7.2:

Proof. The stationary Bewley allocation $\{\hat{a}_t^B(y^t) = 0, \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$
\frac{\hat{a}_t^B(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \tag{57}
$$

Using the fact that $\hat{a}_t^B(y^t) = 0$ and adding

$$\sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t)\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)\hat{v}_t(z_t)$$

to both sides of (57) yields

$$\begin{aligned} & \sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t)\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)\hat{v}_t(z_t) \\ \geq \;& \hat{K}_t(y^t) + \sum_{z_{t+1}} \hat{q}_t(z_{t+1}|z_t)\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)\left[\hat{v}_t(z_t) - \hat{v}_t\right] \\ \equiv \;& \hat{K}_t^*(y^t, z_t) \end{aligned}$$

where $\hat{K}_t^*(y^t)$ is the modified constraint for the de-trended Arrow model. Multiplying both sides by $e_t(z^t)$ gives the modified constraint for the Arrow model with growth stated in the main text.

For the alternative constraint we know that the Bewley equilibrium allocation satisfies

$$\hat{a}_t^B(y^t) + \hat{\sigma}_t(y^t)[\hat{v}_{t+1} + \alpha] \geq \hat{M}_t(y^t). \tag{58}$$

Again using $\hat{a}_t^B(y^t) = 0$ and adding

$$\hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1}(z_{t+1}) + \alpha\right]$$

to both sides of (57) yields

$$\begin{aligned} & \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1}(z_{t+1}) + \alpha\right] \\ \geq \;& \hat{M}_t(y^t) + \hat{a}_t(y^t, z_{t+1}) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1}(z_{t+1}) - \hat{v}_{t+1}\right] \\ \equiv \;& \hat{M}_t^*(y^t, z_{t+1}) \end{aligned}$$

Multiplying both sides by $e_{t+1}(z^{t+1})$ again gives rise to the constraint stated in the main text. □

- Proof of Lemma A.1

Proof. The first order conditions for the representative agent are given by:

$$\begin{aligned} 1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \;\forall z_{t+1} \\ 1 &= \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \end{aligned} \tag{59}$$

□

Proof of Proposition A.1:

Proof. First, we divided through by $e_t(z^t)$ on both sides in equation (A.4):

$$\begin{aligned} \frac{V_t(s^t)}{e_t(z^t)} &= \left[(1-\beta)\frac{c_t^{1-\rho}}{e_t^{1-\rho}} + \beta\frac{(\mathcal{R}_t V_{t+1})^{1-\rho}}{e_t^{1-\rho}}\right]^{\frac{1}{1-\rho}} \\ \hat{V}_t(s^t) &= \left[(1-\beta)\hat{c}_t^{1-\rho} + \beta(\frac{\mathcal{R}_t V_{t+1}}{e_t})^{1-\rho}\right]^{\frac{1}{1-\rho}}. \end{aligned} \tag{60}$$

Note that the risk-adjusted continuation utility can be stated as:

$$\begin{aligned} \frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(E_t\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha}\frac{V_{t+1}^{1-\alpha}}{e_{t+1}^{1-\alpha}}\right)^{1/1-\alpha} \\ &= \left(\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha} \end{aligned}$$

Next, we define growth-adjusted probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}} \text{ and } \hat{\beta}(s_t) = \beta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}.$$

and note that:

$$\begin{aligned} \frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha} \\ &= \left(\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{1/1-\alpha} \hat{\mathcal{R}}_t\hat{V}_{t+1}(s_{t+1}) \end{aligned}$$

Using the definition of $\hat{\beta}(s_t)$:

$$\hat{\beta}(s_t) = \beta\left(\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

we finally obtain the desired result:

$$\hat{V}_t(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t\hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

As before, if the z shocks are ii.d, then $\hat{\beta}$ is constant. □

Proof of Proposition A.2:

Proof. First, we suppose the borrowing constraints are not binding, which is the easiest case. Assume the equilibrium allocations only depend on y^t, not on z^t. Then conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy, for the contingent claim and the stock respectively, read as follows:

$$1 = \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad \forall z_{t+1}$$

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \tag{61}$$

$$* \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad \textit{for all } z_{t+1}. \tag{62}$$

In the first Euler equation, the only part that depends on z_{t+1} is $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ which therefore implies that $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ cannot depend on z_{t+1} : $\hat{q}_t(z^t, z_{t+1})$ is proportional to $\hat{\phi}(z_{t+1})$. Thus define $\hat{R}_t^A(z^t)$ by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A(z^t)} \tag{63}$$

as the risk-free interest rate in the stationary Arrow economy. Using this condition, the Euler equation simplifies to the following expression:

$$1 = \hat{\beta}\hat{R}_t^A(z^t) \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \tag{64}$$

$$\left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \tag{65}$$

Apart from $\hat{R}_t^A(z^t)$ noting in this condition depends on z^t, so we can choose $\hat{R}_t^A(z^t) = \hat{R}_t^A$. □