Insurance and Reserves Management in a Model of Sudden Stops

Ricardo J. Caballero — MIT and NBER
Stavros Panageas — MIT

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PRELIMINARY AND INCOMPLETE

Abstract

Emerging market economies depend on external capital inflows and, for the same reason, are vulnerable to their volatility. In their attempt to smooth the impact of this volatility, countries build large war-chest of international reserves and incur in costly precautionary recessions. In order to assess the efficiency of these smoothing mechanisms and their alternatives, we develop a dynamic model that captures some of the key tradeoffs and constraints faced by these countries in their relation to international financial markets. We estimate the main parameters using a panel of emerging market economies, and simulate alternative scenarios. Aside from the model's ability to generate realistic numbers and scenarios, our main substantive conclusion is that the standard build-up of (non-contingent) reserves is an expensive and imperfect mechanism to hedge capital flow volatility. Instead, countries should seek to hoard assets and issue liabilities that are contingent on variables correlated to these sudden stops and exogenous to them, such as interest rates in the developed world, volatility and "risk appetite" indices, and commodity prices.

JEL Codes: E2, E3, F3, F4, G0, C1.

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1 Introduction

Emerging market economies depend on external capital inflows and, for the same reason, are vulnerable to their volatility. While in many circumstances the behavior of capital flows simply amplify domestic deficiencies, there is extensive evidence that in many others the main culprit is not the country itself but the international financial markets' response to shocks only vaguely related to the country's actions.

The real costs of this volatility for countries that experience open crises are dramatic and well known. However, the costs are also large for prudent economies that do not fall into open crises but are forced to build large war-chests of international reserves and incur in precautionary recessions at the first sight of sudden stop risk. Are there potentially less costly prudential mechanisms to deal with capital flow volatility? Who would be the countries' counterpart in these mechanisms? What is the specific role of reserves accumulation in dealing with capital flows? What kind of instruments should these reserves be invested in? How are these mechanisms and instruments limited by financial and collateral constraints?

These are among the most pressing questions for policy-makers and researchers in emerging market economies and the international financial institutions. Unfortunately, while there has been significant progress over the last two decades in understanding some of the limitations of financial contracting with emerging markets, as well as on the role of reserves in smoothing the impact of capital flow volatility, we are far from having an integral view on the answers to these questions. In this paper we take a step in that direction by developing a framework that simultaneously considers some of the main financial layers and constraints involved in emerging markets external financing. The framework is dynamic and can be used to gauge quantitative answers, which we do.

The framework considers three type of agents: An emerging market country (representative agent), specialist investors, and the world capital markets at large. The essence of an emerging market economy is that its future income is higher than current income so it wishes to borrow. However, the country has great difficulty in pledging future income. Specialists can alleviate this problem but they themselves are subject to shocks that limit their ability to commit to deliver resources. These shocks, which in our model are driven by a Poisson process, are the sudden stops of capital inflows. That is, episodes when specialists are unable to rollover all their explicit or implicit short term commitments.

The country would like to insulate itself from these sudden stops, but it cannot do so with its specialists since they are constrained during these events. Resorting to the world capital markets after the sudden stop takes place does not work either, since the country has very limited credibility with non-specialists. The option that is in principle open, in that it is consistent with the different commitment and informational constraints, is for the country to pay upfront for an insurance policy with world capital markets. The country pays a premium prior to a sudden stop and receives in exchange a flow of transfers while in the sudden stop.

An alternative interpretation of this arrangement, is for the country to buy put options that pay when the sudden stop takes place, and the payoff is used to buy a sort of "annuity" that yields a constant return while the sudden stop runs its course and zero afterwards.

Holding non-contingent international reserves, as central banks do in practice, is strictly dominated by the put-options/annuities strategy. On one hand, prior to the sudden stop, reserves require larger consumption sacrifices while their stock is built. This is particularly costly for countries that suffer tight borrowing constraints. On the other, annuities insure the duration of the sudden stop while reserves do not. But regardless of whether annuities exist or not, reserves will be dominated as long as the put options payoffs are tightly synchronized with sudden stops. Thus to justify holding reserves, one must introduce some "friction" in these put options markets. We study this case as well. For different degrees of precision in the put options markets, we describe the optimal reserves build up prior to sudden stops and their use during these events.

Regardless of the particular composition of reserves and hedging instruments, the incentive to accumulate them is not constant over time. Sudden stops are are not totally unpredictable. In addition to endogenous domestic problems, which are not the focus of this paper, global conditions and factors change over time, introducing time variation in the likelihood of sudden stops. This creates demand for yet another hedging instrument as, ex-post, the cost of put options against sudden stop (or the cost of the credit lines) rises with the likelihood of sudden stops. This increase in the premium, coupled with the increased incentive to accelerate the accumulation of reserves as the sudden stops nears, induces costly precautionary recessions. Thus, hedging against the global factors that raise the likelihood of sudden stops helps smoothing consumption cycles *prior* to sudden stops.

A very similar phenomenon arises with terms of trade shocks. Even if sudden stops are totally unpredictable, the cost of these events differs significantly if at the time they take place the country's terms of trade are high or low. For this reason, an emerging market economy has a significantly stronger incentive than a developed economy to hedge terms of trade shocks even if these are largely transitory and independent of sudden stops. Moreover, if sudden stops are somewhat predictable, the incentive to hedge terms of trade shocks also rises with the likelihood of a sudden stop.

We use a panel of 13 emerging market economies for the period 1980-2002 to gauge the key parameters of the model, and then simulate different scenarios. The numbers generated by our model when countries use no contingent markets —which is more or less the standard practice— are roughly consistent with the observed statistics on reserves accumulation and impact of sudden stops. Aside from the model's ability to generate realistic numbers and scenarios, our main substantive conclusion is that the standard build-up of (non-contingent) reserves is an expensive and imperfect mechanism to hedge capital flow volatility. Instead, countries should seek to hoard assets and issue liabilities that are contingent on variables correlated to these sudden stops and exogenous to them, such as developed world interest rates, volatility and "risk appetite" indices, and commodity prices.

Literature review. MISSING

Section 2 introduces the agents, describes the financial and collateral constraints faced by the country, develops a benchmark model without sudden stops, and concludes by introducing our concept of a sudden stop. Section 3 is the core of the paper. It develops a model with sudden stops and discusses optimal consumption, reserves accumulation and portfolio policies under different degrees of imperfection of hedging markets. Section 4 extends the previous model to allow for a time varying risk of sudden stops. As the sudden stops become more likely, the cost of the precautionary measures rise, causing a sort of precautionary recessions. This generates demand for yet another hedging instrument, one that facilitates smoothing *prior* to sudden stops. We derive initial wealth conditions that interact with the financial constraints faced by the country in determining the extent to which substantial smoothing prior to a sudden stop is feasible for the country. Section 5 extends the model to reduce the extent to which reserves can be used as collateral and to allow for transitory terms of trade shocks. None of the main qualitative conclusions change with the former extension, but this introduces yet another reason for why reserves are dominated by alternative contingent strategies. The latter extension, on the other hand, shows that there is a strong interaction between terms of trade shocks and sudden stops and their impact on hedging demands, even if these shocks are independent of each other. Section 6 estimates the key parameters of the model and quantifies the effects described in the theory. Section 7 concludes and is followed by an extensive appendix that contains all the formal proofs.

2 The Environment and Sudden Stops

While there are many important issues that arise from decentralization in economies with poor institutional development, we leave these aside and focus on problems between the country as a whole and international investors. We study a representative agent economy with a benevolent government that seeks to maximize the expected present value of utility from consumption:

$$\mathrm{E}\left[\int_t^\infty u(c_s)e^{-r(s-t)}\,ds\right]$$

with r being both the discount and riskless interest rate. While it is not essential, assuming a CRRA utility of consumption also simplifies the exposition:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

2.1 The Benchmark: Emerging Markets, Specialists, and World Capital Markets

There are two features of an emerging market economy that are important for our analysis. First, its current income is low relative to its future income (it has yet to catch up), and thus it would like to borrow and run current account deficits. Second, it has difficulty pledging future income to finance these deficits.

We capture the path-of-income feature with a simple income process that takes value zero until a random time τ^G and $Y > 0$ thereafter.[1] We refer to times prior to τ^G as the pre-development phase and times after τ^G as the post-development phase. The focus of the paper is on the former phase. In order to eliminate inessential time dependency, we assume that τ^G is governed by a Poisson process with constant hazard g.

A country in the pre-development phase would like to borrow against its post-development income. We split potential financiers into world capital markets at large (WCM), and specialists. The former have no ability or information to induce the country to repay any amount, while the latter do. Specialists are those investors that have developed some expertise and connections in the country and can reduce the extent of its financial constraint. They accept pledges up to a share z of post development income Y.

In aggregate —that is after netting out the multiple type of financial contracts that individuals may sign and are not of our concern in this paper — specialists optimally engage in "swap-like" contracts with the country. At each time t, the specialists commit to provide resources for $y\,dt$ over the next infinitesimal time interval dt if τ^G does not arrive in that time, in exchange for receiving a perpetuity that has values zY/r if τ^G does arrive, which occurs with "probability" $g\,dt$. If specialists are competitive and risk neutral, which we assume:[2]

$$y = gzY/r$$

By constantly renewing such contracts, the country can guarantee itself a constant income y until τ^G and $(1 - z)\,Y$ thereafter.

We capture the financially constrained aspect of emerging market economies with the assumption:

Assumption 1 *(Limited Unsecured Borrowing)* $z < \frac{r}{r+g}$

which ensures that the funds provided by specialists before development are less than the unpledged income after development. In short: $y < (1 - z)Y$.

[1] In the quantitative section we add a positive pre-development GDP. Assuming this is zero here and interpreting Y as the difference between post- and pre-development GDP simplifies the notation.

[2] Formally, this expression can be derived from the pricing equation:

$$0 = -y + rP + g(zY/r - P)$$

and noting that a swap at the time of its inception has value $P = 0$.

In addition, a country can supplement the resources provided by the specialists with any other financial asset it may have accumulated, X_t, which we call reserves for short. These reserves are composed of international financial instruments so they do not require specialists knowledge and can therefore be pledged as collateral to WCM.

Assumption 2 *(Collateral) Reserves X_t can be pledged as collateral to WCM.*

Similarly to the optimal contracts with specialists, here as well, in aggregate the collateralized contract with WCM takes the form of a swap (secured by reserves). At each time t, risk-neutral and competitive WCM commit to pay $g\overline{X}_t\,dt$ over the next infinitesimal time interval dt if τ^G does not arrive in that time interval, in exchange for receiving $\overline{X}_t \leq X_t$ if τ^G does arrive.

Before introducing sudden stops, let us set up and characterize our benchmark model. Formally, a country that has reserves $X_t \geq 0$ at time t, faces the following problem:

$$
\begin{aligned}
V(X_t) &= \max_{c_s, \overline{X}_s} E\left[\int_t^{\tau^G} e^{-r(s-t)}u(c_s)ds + e^{-r(\tau^G - t)}V^G(X_{\tau^G} - \overline{X}_{\tau^G})\right] \\
& \quad s.t. \\
dX_t &= \left[rX_t + g\overline{X}_t - c_t + y\right]dt \\
\overline{X}_t &\leq X_t \\
y &= gz\frac{Y}{r} \\
X_t &\geq 0 \text{ for all } t \geq 0 \\
\lim_{t\to\infty} e^{-rt}X_t &= 0
\end{aligned}
$$

In the main text, we focus on the case where the country is always constrained before development —that is, it wants to transfer as much resources as possible from the post- to the pre-development phase— and hence:

$$\overline{X}_t = X_t.$$

We shall assume that this constraint binds throughout and study the possibility of other regions in the appendix.

Note also that since the discount and interest rates coincide, and there is no uncertainty after development takes place, it is straightforward to verify that consumption will be constant in the development phase:

$$c_t = (1-z)Y \quad \text{for all } t > \tau^G$$

and that the value function in this pahse is constant and equal to:

$$V^G = \left(\frac{1}{r}\right)^{\gamma} \frac{((1-z)Y/r)^{1-\gamma}}{1-\gamma}.$$

More importantly for what follows, prior to the arrival of the development phase, the country will have *no incentive to accumulate reserves*:

Proposition 1 *For any* $X_0 \geq 0$*, then* $X_t = X_0$ *for all* $t \geq 0$ *and* $c_s = y + (r+g)X_0$ *for all* $0 \leq s < \tau^G$*.*

We relegate the formal proof to the appendix but the intuition behind it is straightforward. First, since the agent is constrained in the pre-development phase (and this constraint is time invariant) any additional accumulation of reserves cannot come from increased borrowing against post-development resources. This means that building reserves entails a reduction in current consumption. Second, since the country pledges all its reserves to WCM, it effectively earns a return of $r + g$ per unit of reserves. This matches exactly the effective discount rate of the agent with respect to reserves — the discount rate r plus the rate g at which development arrives and the pledged reserves must be forfeited. This means that the optimal consumption path is flat. Combining these two features it becomes apparent that the country has no incentive to build or deplete reserves in the pre-development phase, for it only then can ensure a flat consumption path.[3]

It follows, by backward induction, that a country would never build a stock of reserves in this benchmark case. This will provide a useful reference to isolate the nature of any reserves buildup in the next sections.

2.2 Sudden Stops and Insurance Demand

Let us now modify our benchmark economy to introduce the sudden stops shocks that are the main concern of this paper.

In our basic setup above, specialists have unlimited resources and credibility. The country's borrowing constraint limits the amount of smoothing between the pre- and post-development phases, but capital flows remain stable in the former. We introduce sudden stops in our model by assuming that there are episodes when specialists' ability to pledge resources is significantly reduced. Since we are primarily concerned with the country's mechanisms to smooth specialists' shocks, we do not specify the particular agency problem or capital constraints that afflict specialists during sudden stops, but simply state the outcome:

Assumption 3 *(Sudden Stops) During sudden stops, specialists (collectively) can transfer at most* $y^{SS} \equiv$

[3] But wouldn't the country have an incentive to accumulate reserves in order to build collateral and relax the financial constraint? The answer is clearly no. The logic behind this natural question is built on the implicit assumption that collateral is self-financing, in the sense that reserves can be built from additional "borrowing" on the commitment that the reserves so built will be pledged back to those that lent for that purpose. But this cannot be done without violating the commitment limits we have stated for the country. At the outset, the country is still bound by its maximum pledgable income zY. If it wants to build reserves, it cannot expand the amount of loans. It instead must sacrifice the consumption that those loans would have financed. But this is precisely what we have argued above to be sub-optimal.

$gz^L Y/r < y$ resources to the country. The country still trades this lower payoff for zY/r when it develops.[4] *Specialists transit from the normal to the sudden stop stage with hazard λ_t and do the reverse with hazard $\tilde{\lambda}$, both of which are independent of the transition to development.*

The presence of sudden stops effectively means that the flow of resources received from the specialists before development is no longer constant but it drops to y^{SS} during sudden stops. This can be interpreted as a drop in the share of swaps rolled over by the specialists.

If the country takes no precautions, then its consumption drops by $(y - y^{SS}) = gY(z - z^L) > 0$ during sudden stops. Since this is clearly suboptimal, the country seeks to smooth consumption by resorting to WCM or by doing self insurance. We turn to these options next.

3 Sudden Stop Insurance and Reserves

Let us assume for now that the risk of a sudden stop is time-invariant, $\lambda_t = \lambda$.

3.1 World Capital Markets and Sudden Stop Insurance

3.1.1 Perfect Sudden Stop Insurance

World capital markets cannot replace specialists ex-post since they have no mechanism to force the country to repay (for them, $z = 0$). This is the reason they only issue secured (by reserves) swaps. However, as long as sudden stops are verifiable, the country can engage in sudden-stop insurance contracts with WCM; that is contracts paid *in advance* by the country (we assume throughout that WCM have no commitment problems).

In particular, the country and WCM can enter a sequence of option type agreements with maturity dt, whereby the country pays a flows $p\,dt$ prior to the sudden stop in exchange for a flow $h\,dt$ from the WCM for as long as the next sudden stop lasts. If WCM are risk neutral (or sudden stops are uncorrelated to world aggregate shocks), they value such claim before a sudden stop at W, which satisfies:

$$(r + \lambda + g)W = p - \frac{\lambda}{r + \tilde{\lambda} + g} h.$$

The first term on the right hand side is the premium received from the country while the second term is the expected liability associated to the insurance policy. If WCM have enough resources to fulfill all the

[4] This assumption means that specialists realize rents (interest rates rise) during sudden stops. Allocating these rents to the country does not alter anything substantive in the results that follow. In that case, the country only has to give $z^L Y < zY$ to the specialists, should τ^G arrive during a sudden stop.

country's insurance need, which we assume, then $W = 0$ in equilibrium, and:

$$p = \frac{\lambda}{r + \widetilde{\lambda} + g} h. \tag{1}$$

At this price, the country finds it optimal to buy as much sudden stop insurance as it needs to smooth consumption.

Proposition 2 *When there exists a frictionless market for sudden-stop insurance contracts, consumption is constant prior to τ^G, regardless of the occurrence of sudden stops.*

Remark 1 *Note that while consumption is smooth prior to development as in the benchmark case without sudden stops, the level of consumption is lower by p. This is because WCM only facilitate smoothing of sudden stops but cannot relax the overall financial constraint prior to development, which only specialists can do. Thus the latter's temporary breakdowns in their ability to relax the overall constraint is, on average, costly for the country even if it does not introduce volatility prior to development.*

3.1.2 SS-Hedges

These sudden stop insurance contracts with WCM can also be thought of as the combination of two financial instruments: a *sudden stop hedge* (ss-hedge) and a sudden stop *annuity* (ss-annuity). The ss-hedge costs λ per unit time, per dollar received when the sudden stop takes place. The ss-annuity insures the duration of the sudden stop and transforms each dollar paid at the onset of the sudden stop into a flow of $(r + \tilde{\lambda} + g)$ dollars while the sudden stop lasts. Thus the country replicates the above contract by buying ss-hedges to deliver p/λ dollars at the outset of the sudden stop, which are immediately converted into ss-annuities. Since the sudden stop insurance is replicated by this combination, Proposition 2 applies in this case as well.

This decomposition is useful for understanding what is behind the next proposition. Sofar in this section we have not allowed the country to accumulate reserves. As we now show, this is without loss of generality. As long as there is an ss-hedging market, regardless of whether an ss-annuity market exists or not, the country will not accumulate reserves. The reason is that –similar to our benchmark case– borrowing to build reserves is costly because the borrowing constraint binds and hence it sacrifices consumption smoothing prior to the next sudden stop. Instead, purchasing an ss-hedge is a more cost effective mechanism to transfer resources into the next sudden stop, since it allows consumption smoothing prior to that event.

Proposition 3 *If there exists a frictionless ss-hedging market, then i) there will be no drop in consumption at the time of the sudden stop; ii) consumption will be smooth ($dc_t = 0$) prior to a sudden stop, and iii) the country will accumulate no reserves prior to the sudden stop ($dX_t = 0$).*

We relegate to the appendix a more detailed proof of this proposition, however it is instructive to go over the main steps of the argument here since these repeat in most of our analysis.

The Bellman equation of the country prior to a sudden stop is:

$$
\begin{aligned}
0 &= \max_{c,\xi,\overline{X}} \left\{ u(c_t) + \lambda V^{SS}(X+\xi) + g\left(V^G(X-\overline{X}) - V\right) + \mathcal{A}V - (r+\lambda)V - \phi(\overline{X} - X) \right\} \\
\mathcal{A}V &= V_X\left(rX + y - \lambda\xi + g\overline{X} - c\right)
\end{aligned}
$$

where $V^{SS}(X+\xi)$ is the Value function once the country enters the sudden stop and ϕ is a Lagrange multiplier associated with the constraint $\overline{X}_t \leq X_t$. Computing the first order conditions yields

$$
\begin{aligned}
u'(c_t) &= V_X && (2) \\
\lambda V_X^{SS}(X+\xi) &= \lambda V_X(X) && (3) \\
gV_X^G(X-\overline{X}) &= gV_X - \phi && (4)
\end{aligned}
$$

The second of these conditions demonstrates that the country will enter enough contracts ξ to equalize the marginal utilities of consumption right before and after the sudden stop is triggered. That is, there are no drops in consumption at this instant.[5]

In order to see that consumption prior to the sudden stop is constant, it is easiest to think of the optimal control formulation of the problem, for which we have the Hamiltonian (prior to a sudden stop)

$$
\begin{aligned}
H &= e^{-(r+\lambda+g)t}\left[u(c_t) + \lambda V^{SS}(X_t+\xi) + gV^G(X-\overline{X})\right] \\
&\quad + e^{-(r+\lambda+g)t}\left[\mu(rX_t - c_t + y + g\overline{X} - \lambda\xi) - \phi(\overline{X} - X)\right]
\end{aligned}
$$

The first order conditions of this Hamiltonian are identical to (2), (3) and (4), when one defines $\mu_t = V_X$. The dynamic equation for μ_t is

$$
\frac{d\mu_t}{dt} = (r+g+\lambda)\,\mu - \lambda V_X^{SS}(X_t+\xi)g - gV_X^G(X-\overline{X}) - \phi - \mu r
$$

Using the fact that

$$
u'(c) = c^{-\gamma}
$$

and conditions (2) and (4), we obtain

$$
\frac{dc_t}{c_t} = -K_1(X_t)dt
$$

with:

$$
K_1(X_t) = \frac{\lambda}{\gamma}\left(1 - \frac{V_X^{SS}(X_t+\xi)}{V_X(X_t)}\right)
$$

The term $K_1(t)$ captures the incentive to reallocate resources at the margin from a period prior to a sudden stop to the latter. But by first order condition (3), ss-hedging is chosen precisely to equate the value of an

[5] Whether consumption stays constant or declines after the onset of the sudden stop depends on whether an ss-annuity market is available or not. However, this is not our main concern here.

extra dollar in both phases, so $K_1(X_t) = 0$. This shows that consumption is constant prior to the sudden stop.

Finally, in order to show that there is no incentive to accumulate reserves prior to a sudden stop, simply note that by first order condition (2) and the concavity of $V(X)$ (see the appendix), a flat consumption path also implies that reserves remain constant throughout.[6]

3.2 Reserves Accumulation

Up to know we have highlighted the fact that accumulation of reserves is unlikely to be the most desirable precautionary option for an emerging market economy. This is in sharp contrast with actual data, since emerging markets often hold much larger reserves stocks (relative to their size) than developed economies.[7] In practice, the main insurance mechanism used by countries —at least by their central banks— is (non-contingent) reserves accumulation. Part of the reason for this may be simply suboptimal policies, but most likely there are also severe limitations in the markets needed to implement a no-reserves insurance strategy.

Above, we isolated the ss-hedging market as sufficient to eliminate reserves accumulation prior to sudden stops. Thus, in order to find a precautionary role for reserves, we need to introduce some imperfection in the ss-hedging market. Let us start with the extreme case where such market is closed. In this context, the country's only option to reduce the impact of sudden stops is to accumulate reserves. Let us spell out the optimal control problem in full:

[6] Note that our statement is that reserves will not be accumulated prior to a sudden stop. But if no annuity markets exist, then the payoff of the SS-hedge at the onset of the sudden stop will generate reserves at that point. And if the sudden stop is unusually short, this means that the next non-sudden stop period starts with reserves as well. The point of the proposition, however, is that the country does not accumulate reserves as a precautionary mechanism. It says nothing about the particular assets received at the onset of the sudden stop (annuities, bonds, cash or credit lines).

[7] NUMBERS

$$
\begin{aligned}
V(X_t) &= \max_{c_s,\overline{X}_s} E\left[\int_t^{\tau^G\wedge\tau^{SS}} e^{-r(s-t)}u(c_s)ds + e^{-r\left(\left(\tau^G\wedge\tau^{SS}\right)-t\right)}\tilde{V}(X_\tau)\right] \\
V^{SS}(X_t) &= \max_{c_s,\overline{X}_s} E\left[\int_t^{\tau^{NSS}\wedge\tau^G} e^{-r(s-t)}u(c_s)ds + e^{-r\left(\tau^G\wedge\tau^{NSS}-t\right)}\tilde{V}^{SS}(X_\tau)\right] \\
& \quad s.t. \\
dX_t &= \left[rX_t - c_t + g\overline{X}_t + y^{SS}1\{SS\} + y1\{NSS\} + Y(1-z)1\{G\}\right]dt \\
Y &> y = gzY/r > y^{SS} = gz^LY/r \\
X_t &\geq 0 \text{ for all } t \geq 0 \\
\lim_{t\to\infty} e^{-rt}X_t &= 0 \\
\tilde{V}(X_\tau) &\equiv V^G(X_t - \overline{X}_t)1\{\tau^G \leq \tau^{SS}\} + V^{SS}(X_\tau)1\{\tau^G > \tau^{SS}\} \\
\tilde{V}^{SS}(X_\tau) &\equiv V^G(X_t - \overline{X}_t)1\{\tau^G \leq \tau^{NSS}\} + V(X_\tau)1\{\tau^G > \tau^{NSS}\}
\end{aligned}
$$

The indicator function $1\{G\}$ is equal to one after "development arrives," $1\{SS\}$ is the indicator function when the country is currently in a sudden stop, and $1\{NSS\}$ is the indicator function if the country is not in a sudden stop (and has not developed yet).

Proposition 4 *In the absence of ss-hedge markets, the country accumulates reserves prior to sudden stops up to a level $X^* = ((1-z)Y - y^{SS})/(r+g)$, and uses these reserves during the sudden stops. Moreover, if $X < X^*$, consumption will drop at the sudden stop.*

Let us again leave the most technical details of the proof to the appendix and develop only the main steps here. For this, define the Hamiltonian before a sudden stop as:

$$
\begin{aligned}
H &= e^{-(r+\lambda+g)t}\left[u(c_t) + \lambda V^{SS}(X_t) + gV^G(X-\overline{X})\right] \\
&\quad + e^{-(r+\lambda+g)t}\left[\mu\left(rX_t - c_t + y + g\overline{X}\right) - \phi(\overline{X} - X)\right]
\end{aligned}
$$

where ϕ is a Lagrange multiplier associated with the constraint $\overline{X} < X$. With steps similar to section 3.1 we obtain the first order conditions and the dynamic equation for the co-state variable:

$$
\begin{aligned}
u'(c_t) &= \mu \\
\frac{d\mu}{dt} &= (r+\lambda+g)\mu - \left[\lambda V_X^{SS}(X_t) + \mu r + gV_X^G\left(X-\overline{X}\right)\right] - \phi
\end{aligned}
$$

Using the fact that $u'(c_t) = c_t^{-\gamma}$, and replacing the equation of the co-state variable in the resulting expression shows that the optimal solution prior to the sudden stop satisfies the following system of (non-linear) ODE's:

$$
\begin{aligned}
\frac{dc_t}{c_t} &= -K_1(X_t)dt \\
dX_t &= \left(rX_t - c_t + y + g\overline{X}_t\right)dt
\end{aligned}
$$

where:

$$K_1(X_t) = \frac{\lambda}{\gamma}\left(1 - \frac{V_X^{SS}(X_t)}{V_X(X_t)}\right)$$

This is similar to the expression we found in the case where ss-hedges existed, but the key difference is that in the absence of the latter the value of an extra dollar is worth more in the sudden stop than prior to it as long as $X < X^*$ (which is nearly always the case, see Section 6). Thus $K_1(X_t) < 0$ and consumption is increasing over time. But since $u'(c_t) = V_X(X_t)$ and $V(X_t)$ is concave, an increasing consumption path also implies an increasing path of reserves.

Now turning to the sudden stop phase, similar steps yield:

$$\frac{dc_t}{c_t} = -K_1^{SS}(X_t)$$

where

$$K_1^{SS}(X_t) = \frac{\tilde{\lambda}}{\gamma}\left(1 - \frac{V_X(X_t)}{V_X^{SS}(X_t)}\right)$$

Observe that now $K_1^{SS}(X_t) > 0$ by the same argument used above (a marginal dollar is worth more in the sudden stop than prior to it), which implies that consumption is decreasing during the sudden stop. But since $u'(c_t^{SS}) = V_X^{SS}(X_t)$, a decreasing consumption path and the concavity of V^{SS} imply that reserves are being depleted during the sudden stop.[8]

Putting things together, we conclude that the country sacrifices consumption smoothing prior to the sudden stop in order to build reserves and smooth the impact of sudden stops on consumption. In general, however, consumption will drop at the sudden stop. That is, this mechanism of smoothing sudden stops is both expensive and incomplete.

3.3 An Intermediate Case: Imperfect SS-hedges

Neither of the extremes highlighted in the previous sections is likely to be an accurate description of the situation faced by most countries. On one hand, the perfect ss-hedges of the sort described in Section 3.1 are not available. On the other, holding reserves exclusively is likely to be suboptimal if there are assets that exhibit somewhat predictable reactions during a sudden stop.

Let us start our discussion of imperfect ss-hedges by assuming that there exist options on traded securities, with a price that jumps in tandem with the sudden stops.[9] In order to keep the analysis comparable with section 3.1 we shall assume that the country engages in a flow of short term contracts that require a flow

[8] Note as well that for sufficiently long sudden stops, reserves will be entirely depleted. To see this, note that the ratio $V_X(X_t)/V_X^{SS}(X_t)$ is increasing with respect to X_t, so that as reserves fall during the sudden stop, the incentive to use them rises.

[9] One such variable in practice is the Volatiliy Index, VIX, which is available in the US since 1986. We return to this variable later in the paper.

payment $\psi\, dt$. At each point in time, if no sudden stop occurs the most recent contract expires "out of the money." However, if a sudden stop does occur, the contract pays out a given (positive) but random amount ζ with mean $\overline{\zeta}$. The randomness in the payoff captures the imperfect nature of the ss-hedge.

Formally, the budget constraint now becomes

$$dX_t = \left(rX_t - c_t + y + g\overline{X}_t - \xi_t\psi\right) dt + \xi_t\zeta dN_t$$

where ξ are the contracts purchased and dN_t takes a value of zero if no sudden stop takes place and one if it does. Once a sudden stop takes place, ζ is drawn from a distribution $F(\zeta)$ with mean $\overline{\zeta}$ and support in $(0, \infty)$. Since WCM are risk neutral, ψ is simply given by:

$$\psi = \lambda\overline{\zeta} \tag{5}$$

Proposition 5 *When $F(\zeta)$ is non-degenerate, there is an expected rise in the marginal utility of consumption at the time of the sudden stop. Moreover, in this case non-contingent reserves are accumulated prior to sudden stops and used during these events.*

The proof is developed in the Appendix, however we highlight here the steps needed to show the connection between the imperfect nature of the ss-hedge (non degenerate $F(\zeta)$), and the expected drop in consumption at the time of the sudden stop. The statements on reserves then follow from the same reasoning used in the previous propositions: If the value of a marginal dollar is higher in the sudden stop than prior to it, which occurs when consumption is lower at the outset of the sudden stop than right before it, then there is an incentive to accumulate reserves and use them to smooth the sudden stop.

In order to show that on average consumption drops at the sudden stop, note that the Bellman equation prior to this event is now:

$$\begin{aligned} 0 \;=\;& \max_{c_t} \{u(c_t) - c_t V_X\} + \max_{\overline{X}_t} \left\{gV^G(X_t - \overline{X}_t) + gV_X\overline{X} - \phi(\overline{X} - X)\right\} + \\ & + \max_{\xi_t} \left\{\lambda \int_0^\infty V^{SS}(X_t + \xi_t\zeta)dF(\zeta) - \xi_t\psi V_X\right\} \\ & + V_X\left(rX_t + y\right) - (r + \lambda + g)V(X_t) \end{aligned}$$

with first order conditions:

$$u'(c_t) = V_X \tag{6}$$

$$\lambda \int_0^\infty V_X^{SS}(X_t + \xi_t\zeta)\zeta dF(\zeta) = V_X\psi$$

$$gV_X^G(X - \overline{X}) = gV_X - \phi \tag{7}$$

Combining the latter with the equilibrium price in (5) yields:

$$\int_0^\infty V_X^{SS}(X_t + \xi_t\zeta)\left(\frac{\zeta}{\overline{\overline{\zeta}}}\right) dF(\zeta) = V_X \tag{8}$$

Note that as long ζ has a non-degenerate distribution, then :

$$E\left(\frac{V_X^{SS}(X_t+\xi_t\zeta)}{E\left[V_X^{SS}(X_t+\xi_t\zeta)\right]}\left(\frac{\zeta}{\overline{\overline{\zeta}}}\right)\right)<1 \tag{9}$$

which when replaced in (8) implies that:

$$1=\frac{E\left(V_X^{SS}(X_t+\xi_t\zeta)\left(\frac{\zeta}{\overline{\overline{\zeta}}}\right)\right)}{V_X}<\frac{E\left[V_X^{SS}(X_t+\xi_t\zeta)\right]}{V_X} \tag{10}$$

and thus from the first order conditions of optimality:

$$E(u'(c_{\tau^{SS+}}))=E(V_X^{SS}(X_t+\xi_t\zeta)>V_X\left(X_t\right)=u'(c_{\tau^{SS-}})$$

The reason for this result is that agents are afraid of “over-committing” to a hedge that may backfire or deliver sub-par returns when resources are needed the most. The flip side of this is that the country will hold some (non-contingent) reserves, which by their high opportunity cost are seldom accumulated in sufficient amount to completely smooth the sudden stop. It is apparent that both the expected rise in marginal utility of consumption and the amount of non-contingent reserves accumulation are decreasing with respect to the precision of the ss-hedge.

Finally, it is interesting to note that the imperfection in the ss-hedge needs not come from its uncertain payoff but it could also be the result of some market imperfection or risk premium that adds a markup to the price of the hedge, so that (5) is replaced for:

$$\psi>\lambda\overline{\zeta} \tag{11}$$

It is straightforward to show that in this case the country will curtail its purchase of ss-hedges, accumulate some reserves, and experience a drop in consumption when hit by a sudden stop.

4 Time Varying Risk

In practice not only there are variables and assets that are contemporaneously correlated with sudden stops, as in the previous section, but also sudden stops are somewhat predictable. In addition to endogenous domestic problems, which are not the focus of this paper, global conditions and factors change over time, introducing time variation in the likelihood of a sudden stop. We capture this feature by introducing a signal variable, s_t, that follows a diffusion process:

$$ds_t=\mu(s_t)dt+\sigma(s_t)dB_t$$

and let the sudden stop hazard depend on this signal:

$$\lambda(s_t) > 0 \qquad \lambda'(s_t) > 0.$$

For simplicity, we keep the hazard to exit the sudden stop fixed at $\tilde{\lambda}$.

The basic impact of this ingredient on decisions prior to the sudden stop can be readily seen in the scenario where ss-hedging and annuity markets function perfectly (and there are no reserves). Recall that in this context consumption is completely smooth prior to a sudden stop if λ is constant. But this is no longer the case here since the price of a unit of ss-hedge is now $\lambda(s_t)$, which is time varying. Naturally, as the likelihood of a sudden stop rises, the price of hedging such event rises. Given that the country is borrowing-constrained, paying this higher premium requires for consumption to fall. That is, the country incurs in a *precautionary (consumption) recession* as the sudden stop signal deteriorates.

4.1 Reserves accumulation and precautionary recessions

If we now remove the possibility of ss-hedging markets (and, less importantly, ss-annuities) the impact of this time varying risk of sudden stops is to introduce fluctuations in the country's incentive to accumulate reserves and hence to sacrifice consumption smoothing prior to a sudden stop. As the risk of a sudden stop rises, the country reduces consumption and increases reserves' accumulation.

Proposition 6 *Whenever $\lambda'(s_t) > 0$, the consumption process is a diffusion with non-zero volatility.*

Note that the Bellman equation prior to the sudden stop is now

$$0 = \max_{c, \overline{X} \leq X_t} \left\{ u(c_t) + \lambda(s_t) V^{SS}(X, s) + g\left(V^G(X_t - \overline{X}) - V(X, s)\right) - (r + \lambda(s_t))V(X, s) + A^R(V(X, s)) \right\} \tag{12}$$

where

$$A^R(V(X, s)) = V_X(X, s)\left(rX_t + g\overline{X} + y - c\right) + \mu(s_t)V_s(X, s) + \frac{1}{2}\sigma(s_t)^2 V_{ss}(X, s) - \phi(\overline{X} - X)$$

The first order condition of this problem yields (where we suppress the value function dependence on X and s, when it does not lead to confusion)

$$u'(c_t) = V_X \Rightarrow c_t = V_X^{-\frac{1}{\gamma}}$$

which, by Ito's Lemma, implies

$$\frac{dc_t}{c_t} = Adt - \frac{\sigma(s_t)}{\gamma}\frac{V_{Xs}}{V_X} dB_s \tag{13}$$

In the appendix we show that A is given by

$$A = -\frac{1}{\gamma}\lambda(s_t)\left(1 - \frac{V_X^{SS}}{V_X}\right) + \frac{1}{2}\left(\frac{V_{Xs}\sigma(s_t)}{V_X}\right)^2\left(\frac{\gamma+1}{\gamma^2}\right)$$

Note first that A is positive and reflects two precautionary savings effects. The first term captures precautioning against a sudden stop similar to Section 3.2. The second one reflects precautioning due to the diffusion in the consumption process. Finally note that since V_{Xs} is positive (an extra dollar is worth more if a sudden stop is more likely), the diffusion term in expression (13) shows that as the signal deteriorates, consumption declines. This is precisely the concept of precautionary recessions we wish to highlight.

Note that the same logic applies to the speed at which the country uses its reserves during the sudden stop phase. The derivations are similar and hence we arrive to a consumption process during sudden stops described by:

$$\frac{dc_t^{SS}}{c_t^{SS}} = A^{SS} dt - \frac{\sigma(s_t)}{\gamma} \frac{V_{Xs}^{SS}}{V_X^{SS}} dB_s$$

with

$$A^{SS} = -\frac{1}{\gamma}\widetilde{\lambda}\left(1 - \frac{V_X}{V_X^{SS}}\right) + \frac{1}{2}\left(\frac{V_{Xs}^{SS}\sigma(s_t)}{V_X^{SS}}\right)^2 \left(\frac{\gamma+1}{\gamma^2}\right)$$

Here as well, $V_{Xs}^{SS} > 0$. Even though the signal does not affect the likelihood of exiting the sudden stop, it affects the likelihood of another sudden stop re-occurring soon after the country exits the current one. So an increase in the signal leads to a more prudent use of limited reserves and hence to a sharper decline in consumption.

4.2 $d\lambda$-hedging

The presence of time varying risk introduces an additional source of hedging demands that is distinct from the hedging demands that we have studied sofar. We will label this new hedging demand as "hedging against time varying risk" ($d\lambda$-hedging for short). It stems from the need to smooth the cost of preventive measures prior to the sudden stop.

Let us now introduce the possibility of $d\lambda$-hedging. For this, assume there is an asset with price process F_t that is perfectly (negatively) correlated with s_t. Also assume that it is uncorrelated with worldwide risks so that its drift is equal to the riskless rate:

$$\frac{dF_t}{F_t} = -\sigma_F dB_t \tag{14}$$

The existence of this asset modifies the wealth accumulation equation prior to the sudden stop to:

$$dX_t = ((r+g)X_t + y - c_t)\,dt - \pi_t \sigma_F F_t dB_t$$

for $X_t \geq 0$, where π_t is the number of shares of the asset the country chooses to purchase, and we have set $\overline{X}_t = X_t$. Then, and in contrast to Proposition 6,

Proposition 7 *Whenever there exists an asset with dynamics given by (14), optimal consumption will have a zero diffusion term.*

Let us only develop the argument prior to the sudden stop, which also applies, with a few minor modifications, during the sudden stop. We develop the arguments for the diffusion component only, and leave the derivations for the drift to the appendix. The Bellman equation is

$$0 = \max_{c,\overline{X}\leq X_t} \left\{u(c_t) + \lambda(s_t)V^{SS}(X,s) + g\left(V^G(X_t - \overline{X}) - V(X,s)\right) - (r + \lambda(s_t))V(X,s) + A(V(X,s))\right\}$$

$$\begin{aligned} A(V(X,s)) &= V_X(X,s)\left(rX_t + g\overline{X} + y - c\right) + \mu(s_t)V_s(X,s) + \frac{1}{2}\sigma(s_t)^2 V_{ss}(X,s) - \phi(\overline{X} - X) \\ &\quad + \frac{1}{2}\sigma_F^2 V_{XX} F_t^2 \pi^2 - \pi V_{Xs}\sigma_F F \sigma(s_t) \end{aligned}$$

The first order conditions are

$$\begin{aligned} u'(c_t) &= V_X \\ \pi &= \frac{V_{Xs}}{V_{XX}}\frac{\sigma(s_t)}{\sigma_F F_t} \\ gV_X^G(X_t - \overline{X}) &= gV_X - \phi \end{aligned}$$

Applying Ito's Lemma to the optimal consumption process yields:

$$\begin{aligned} c_t &= V_X^{-\frac{1}{\gamma}} \\ dc_t &= Adt - \frac{1}{\gamma}V_X^{-\frac{1}{\gamma}}\left(-\frac{V_{XX}}{V_X}\pi_t\sigma_F F_t + \frac{V_{Xs}}{V_X}\sigma(s_t)\right)dB_t \end{aligned}$$

where A is determined in the appendix. Now plugging in for π yields:

$$-\frac{V_{XX}}{V_X}\pi_t\sigma_F F_t + \frac{V_{Xs}}{V_X}\sigma(s_t) = -\frac{V_{XX}}{V_X}\frac{V_{Xs}}{V_{XX}}\frac{\sigma(s_t)}{\sigma_F}\sigma_F F_t + \frac{V_{Xs}}{V_X}\sigma(s_t) = 0$$

That is, $d\lambda$-hedging eliminates the diffusion term. This means that even if there are no ss-hedge markets, reserves should not be exclusively held in non-contingent assets. Investing in assets negatively correlated with the sudden stop signal helps smoothing consumption prior to a sudden stop. It does so by yielding high returns precisely when the country has an incentive to accelerate its reserve buildup, in *anticipation* of a more likely sudden stop in the near future.

In the appendix we show (after computing the drift A) that

$$\frac{dc_t}{c_t} = -\frac{\lambda(s_t)}{\gamma}\left(1 - \frac{V_X^{SS}}{V_X}\right)dt$$

In other words, the dynamics of consumption are very similar to the ones developed in section 3.2. However, as might be expected, the drift of c_t depends on s_t both directly (through the hazard rate $\lambda(s_t)$ and indirectly through the dependence of V_X^{SS} and V_X on s_t). [10]

[10] Moreover, it is important to stress that this analysis applies only to sets where $X_t > 0$. On sets where $X_t = 0$ the consumption will also exhibit singular increases of the sort described in He and Pages (1993)

Of course, if we reintroduce ss-hedges then the country can do better with a combination of $d\lambda$-hedging and ss-hedging, and there is in such case no role for non-contingent reserves. The returns on holdings of F_t are used to purchase ss-hedges, whose price move in tandem with these returns. The question arises in this context on how perfectly can this combination work. Could it completely smooth consumption prior to a sudden stop? That is, not only eliminate the diffusion term in the consumption process but also the drift term? The following proposition states that this can indeed happen but only if X_0 is sufficiently large to ensure that X_t may never reach zero before the next sudden stop.

Proposition 8 *Define*

$$\begin{aligned} Z(s) &= E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}\lambda(s_t)V_X^{SS-1}(u'(c_0))ds|s\right) \\ z(s) &= E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}ds|s\right) \end{aligned}$$

If there exist perfect ss-hedging and $d\lambda$-hedging markets, then $dc_t = 0$ whenever $X_t > 0$, Moreover $X_t > 0$ for all $0 \leq t \leq \tau^{SS}$ if either:

$$X_0 \geqslant E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du} y\lambda(s_t)ds|s_0\right) \tag{15}$$

or:

$$X_0 \geqslant \inf_s \left[Z(s)z(s_0) - Z(s_0)z(s)\right] \tag{16}$$

In words, the more comprehensive statement in the proposition says that if initial wealth is sufficient to cover all the expenses that could arise from smoothing the variation in the price of ss-hedging — that is, the cost of all the future $d\lambda$ positions under the worst (most expensive) path possible — we are back into the setup of Section 3.1. Note that condition (16) is satisfied trivially whenever λ_t is constant (the case in Section 3.1). More generally, it is satisfied for arbitrarily low X_0 as long as the range of variation in $\lambda(s_t)$ is small enough.

5 Extensions: Costly Reserves Holdings and Terms of Trade Shocks

5.1 Costly Reserves Holdings

A running argument throughout the paper has been that (non-contingent) reserves accumulation is a relatively expensive and incomplete mechanism for insuring sudden stops. It turns out that the situation is probably worse than we have stated up to now, since we have assumed throughout that reserves can be pledged in full and at no cost. Effectively, this means that there is no cost of holding (as opposed to accumulating) reserves. This is unlikely to be the case in reality, where central banks often complain of the costs associated to maintaining reserves in very low yielding assets. In our model, such cost can be introduced by reducing the extent tow which reserves can be used as collateral.

If instead of Assumption 2 we have:

Assumption 4 *(Imperfect Collateral) A share $0 \leq \alpha < 1$ of X_t can be pledged as collateral to WCM.*

then the qualitative features of all our "anti-reserves" results hold, but the quantitative effects are larger against reserves accumulation. The reason for this extra effect can be easily appreciated in the benchmark model (no sudden stops) and extrapolated from there to the main propositions.

The country's problem in this modified benchmark model is now:

$$
\begin{aligned}
V(X_t) &= \max_{c_s, \overline{X}_s} \int_t^{\tau^G} e^{-r(s-t)} u(c_s) ds + e^{-r(\tau^G - t)} V^G(X_{\tau^G} - \overline{X}_{\tau^G}) \\
& \quad s.t. \\
dX_t &= \left[rX_t + g\overline{X}_t - c_t + y \right] dt \\
\overline{X}_t &\leq \alpha X_t \\
y &= gz\frac{Y}{r} \\
X_t &\geq 0 \text{ for all } t \geq 0 \\
\lim_{t \to \infty} e^{-rt} X_t &= 0
\end{aligned}
$$

As before, we constrain our attention to the region where the country pledges as much of its reserves as it can:

$$\overline{X}_t = \alpha X_t < X_t.$$

Solving the optimal control problem along lines similar to that in the previous section, now yields the consumption process prior to development:

$$\frac{dc_t}{c_t} = -K_2(X_t)$$

where

$$K_2(X_t) = \frac{g(1-\alpha)}{\gamma} \left(1 - \frac{V_X^G((1-\alpha)X_t)}{V_X(X_t)} \right)$$

which is clearly positive since the country is constrained and values an extra dollar more in the pre- than in the post-development phase.

This implies that if a country has positive reserves, it will not stabilize them as it did when $\alpha = 1$, but deplete them. The reason for this is that the reserves that cannot be pledged, $(1-\alpha)X_t$, end up financing post-development consumption if τ^G realizes. This raises the opportunity cost of holding reserves, since not consuming today sacrifices high marginal utility consumption which may end up financing post-development low marginal utility of consumption.

As mentioned above, this mechanism is more general than the particulars of the model, as it represent a situation where the return on reserves is less than the cost of the liabilities issued to build them. This is the standard case in practice, where reserves are maintained in highly liquid short term instrument and are built against costlier long term liabilities.[11]

Returning to our model, it suffices to say that the term $K_2(X_t)$ is to be added to $K_1(X_t)$ and $K_1^{SS}(X_t)$ (replacing V_X for V_X^{SS}) in all the propositions that described the path of consumption and reserves. That is, as α declines, both the incentive to build reserves prior to a sudden stop falls, and these reserves are used at a faster rate once the sudden stop arrives.

5.2 Terms of Trade shocks

MISSING

Even if sudden stops are totally unpredictable, the cost of these events differs significantly if at the time they take place the country's terms of trade are high or low. For this reason, an emerging market economy has a significantly stronger incentive than a developed economy to hedge terms of trade shocks even if these are largely transitory and independent of sudden stops. Moreover, if sudden stops are somewhat predictable, the incentive to hedge terms of trade shocks rises with the likelihood of a sudden stop.

6 A Quantitative Assessment (Preliminary and incomplete)

6.1 Estimation - Calibration procedure

The underlying structure of our economy in the constant sudden stop risk case is characterized by the following parameters: λ, $\widetilde{\lambda}$, y, y^{SS}, α, g, Y, γ and r. The first five of these are central and the other are ancillary. Correspondingly, we estimate the first group and the others we simply set to "reasonable" numbers. In the case of the time-varying risk model, we estimate the function $\lambda(s_t)$.

Starting from the ancillary parameters, we set $r = 0.03$ and $\gamma = 2$. The model is intended to capture capital flows and the capital account but it has no reference to the level of GDP during pre-development. We interpret c and Y in the model as in deviation from this base GDP. In this section we set this base GDP to one and we add it to consumption in the utility function. Thus all the reported numbers can be interpreted

[11] In fact, it is often suggested that reserves in the sense used in this paper (as a precautionary mechanism against capital flow reversals) are only so when net of short term liabilities.

in units of base GDP. In this context, we interpret $g(Y-1)$ as the expected growth rate of emerging market countries, and Y as the ratio of GDP in countries that transit to development relative to those that do not (for example, Australia and Canada relative to Argentina and Venezuela). We set Y to 5 and expected growth to 0.04.

We now turn to the procedure used to identify sudden stops, their duration and frequency. The data are from three different sources: the IMF's International Financial Statistics (IFS) database, the World Bank's World Development Indicators Database (WDI), and the US Federal Reserve's FRED-II database. The data are annual from 1979 to 2002. We use the following series (the source is in brackets): capital flows proxied by the financial account of the balance of payments (from the IFS), the reserve assets account of the balance of payments (also from the IFS), nominal GDP in US dollars (WDI), average nominal exchange rate (in local currency per dollar, from the IFS), GDP deflator (IFS), and terms of trade effect (WDI).[12] In order to estimate $\lambda(s_t)$, we use the following variables as potential sudden stop signals: the corporate spread in the US (defined as the spread between AAA and BAA bonds, from FRED-II), the 10-year Treasury constant maturity rate (FRED-II), the growth rate of the US GDP (FRED-II), the non-fuel commodity price index (IFS), and the oil price (average series from the IFS). We only use global rather than country specific variables to illustrate that there are clearly exogenous variables that could in principle be used as signals.[13]

Our data set contains annual information on all the variables described above for thirteen emerging market economies: Argentina, Bolivia, Brazil, Chile, Colombia, Ecuador, Indonesia, Korea, Malaysia, Mexico, Peru, Thailand, and Turkey. We do not include countries from Eastern or Central Europe as they did not participate in financial markets during the 1980s.

Our empirical strategy has two steps. In the first one we identify the sudden stop episodes based on information on capital flows, terms of trade effects, and reserves. In the second one we estimate the different hazards given the identified sudden stops and transitions.

We identify the sudden stops and their duration with the following algorithm: i) we create a dummy variable that takes a value of one in any year for which the decline in capital flows with respect to the previous one or two years is at least 5 percent of smooth GDP (measured as a 3-year moving average); ii) after the first period of a sudden stop is identified, the algorithm assigns a one to the country until the capital flow level returns to zero or there are at least two observations showing a positive change in the capital flows; iii) in order to separate demand from supply shocks to capital flows, we also require that to identify the beginning of a sudden stop the country must also be losing reserves. Moreover, if the country accumulates foreign reserves for more than 2% of GDP during the sudden stop, then the algorithm considers that the sudden stop has ended; iv) if a year marked with a zero (non sudden stop) is between two ones,

[12] This series measures the income effect of the variation in terms of trade with respect to the base year used in the national accounts.

[13] See Caballero and Panageas (2003) for a hazard that includes country specific variables.

then is reset to one; and v) if the above criteria are borderline we look at quarterly data when we have it. This criterion only affect classifying 1994 as a sudden stop episode for Mexico (the sharp decline in capital flows accelerated toward the end of that year).

We use two measures of capital flows: the financial account, and the financial account plus the terms of trade effect (both as percentage of GDP). The rational for the latter is that in the absence of financial constraints, capital flows would rise when terms of trade decline. Thus a decline in terms of trade not matched by a rise in capital flows raises a flag that the country may be constrained. With these two series and the zeros and ones identifying normal and sudden stop years, we estimate λ, $\tilde{\lambda}$, y and y^{SS}. Starting from the last two, we identify y with the average capital flow in non sudden stop periods, and y^{SS} as y minus the average decline in capital flows during sudden stops. Note that this yields a negative y^{SS} while in the model is positive; this is just a normalization since our model is only designed to capture the change in capital flows and consumption and not their level.

The parameters λ and $\tilde{\lambda}$ are estimated from:

$$\hat{\lambda} = \left[(1/n) \times \sum_{i=1}^{n} t_i\right]^{-1} \tag{17}$$

$$\hat{\tilde{\lambda}} = \left[(1/n) \times \sum_{i=1}^{n} \tilde{t}_i\right]^{-1} \tag{18}$$

where t_i is the duration, in years, of the period between sudden stops i and $i-1$, $\tilde{t}_i$ is the duration of sudden stop i, and n is the total number of sudden stops observed in the sample.

	$SS/(NSS+SS)$ (%)	$\hat{\lambda}$	$\hat{\tilde{\lambda}}$	Size of SS (% of GDP)	CF in no-SS (% of GDP)
without ToT	22.86	0.136	0.417	-6.71	3.30
with ToT	29.59	0.179	0.402	-8.33	4.32

We estimate a parametric hazard function for the time varying sudden stop risk model:

$$\lambda(s) = \exp(x_s\beta)$$

where x_s is a vector that contains the price of oil, US GDP's growth rate, the 10-year Treasury rate, the non-fuel commodity price index, the US corporate spread, country dummies, and a dummy for oil exporting countries. Rather than reporting the individual coefficients, in the next table we report a series of statics for the logarithm of the estimated hazard, evaluated at the mean of all the covariates but the corporate spread which is allowed to vary. The second column shows the same statistics for the corporate spread.

Statistic	$\log(\hat{\lambda})$	Corporate Spread
Maximum	-0.5907	2.0458
Minimum	-2.417	0.5600
Mean	-1.867	1.007
Persistence (ρ)	0.807	0.807
St. Dev. Innovations	0.261	0.212

Finally, in order to estimate α we need to make an assumption on what kind of instruments the country is using to smooth sudden stops. We take the extreme, but not unrealistic, case where all the smoothing is done with non-contingent reserves. We calibrate $\alpha = 0.435$ to yield the average reserves to GDP ratio during non-sudden-stops in our data, 0.099.

6.2 Magnitudes in the absence of hedging

Let us start by presenting the quantitative implications of our model in a context where no hedging instruments or contracts are used. This is an important case since it is the closest to the standard practice in most emerging market economies.

In all the figures below, we describe the behavior prior to a sudden stop (no label or labelled NSS) and during sudden stops (labelled SS). Figure 1 describes the policy functions $C(X)$ and $C^{SS}(X)$ corresponding to consumption given a level of reserves, prior to the sudden stop and during a sudden stop, respectively. The difference between $C(X)$ and $C^{SS}(X)$ is positive and rises as reserves fall. This is intuitive. When reserves are high and the country enters a sudden stop, there is less need for adjustment because reserves can be used to smooth consumption. On the other hand, if reserves are low, there is little smoothing a country can do since the sudden stop is likely to last more than an instant and the limited reserves must be spread through the sudden stop. The above also implies that a long lasting sudden stop will eventually lead to sharp consumption drops even if the initial level of reserves is relatively large. For example, if a sudden stop takes place when the level of reserves is around 20 percent of GDP, the initial consumption drop is only around 2 percent of GDP, but as the sudden stop progresses consumption can fall by more than 9 percent (of base GDP) relative to its pre-sudden stop level.[14]

Associated to these consumption policies, there are transitional dynamics for reserves. Figure 2 plots

[14] It is important to keep in mind in the above figure as well as in all those that follow, that ours is a model of current account deficits, which here translates one-for-one into a model of consumption because we have assumed exogenous income. If we were to add a realistic aggregate demand effect, then the declines in consumption would be significantly larger than the corresponding changes in the current account deficits. NUMBERS.

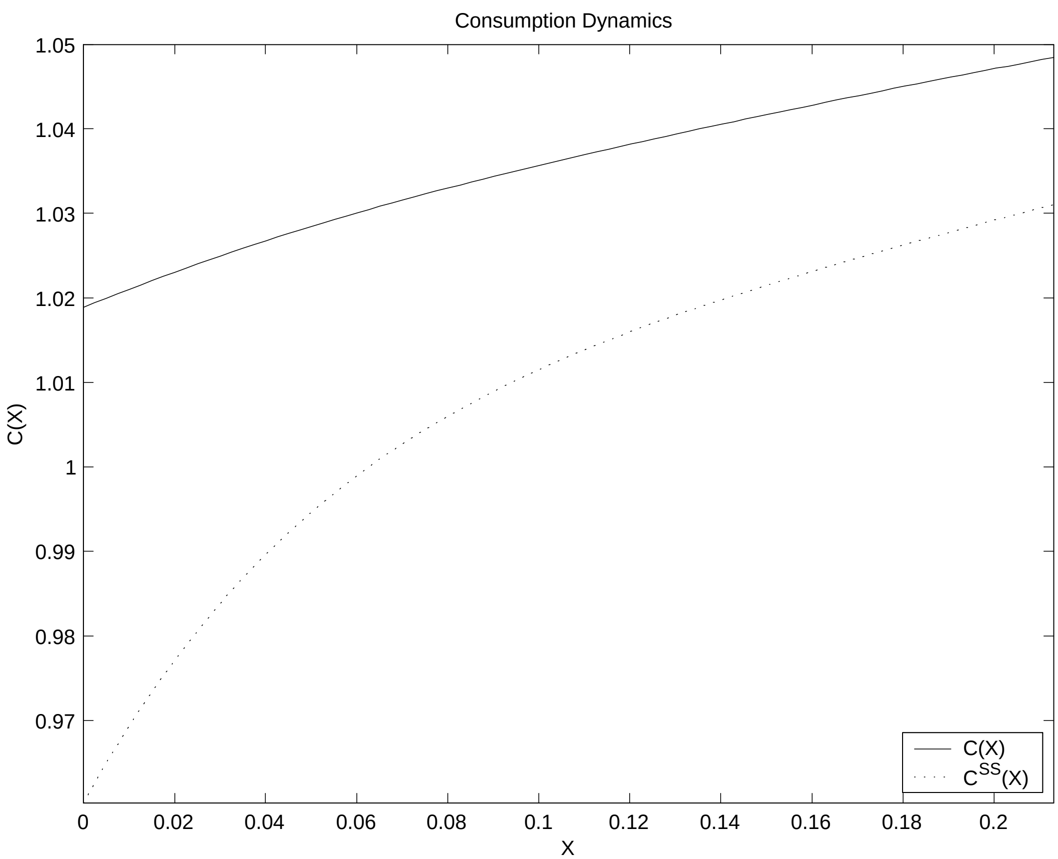

Figure 1: Consumption Dynamics.

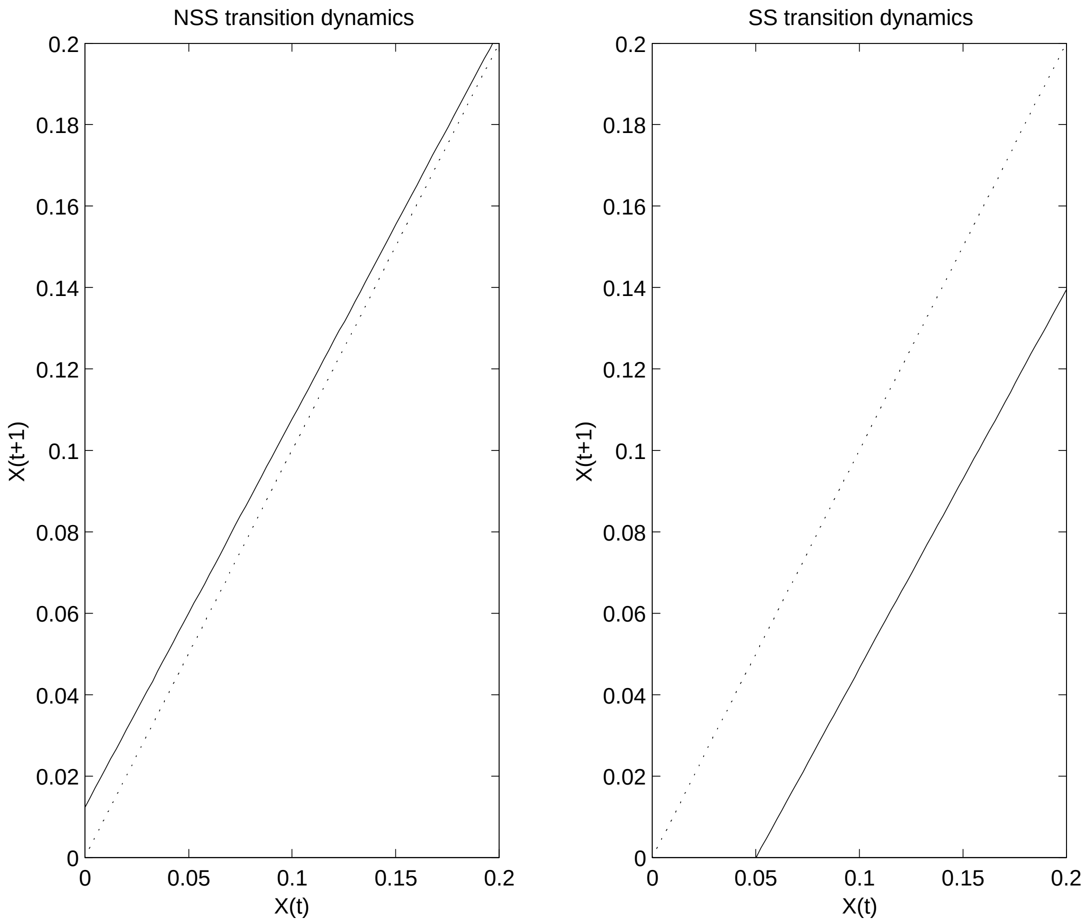

Figure 2: Transitional Dynamics of Reserves.

X_{t+1} as a function of X_t. The dotted line in both panels represents the 45 degree line. The solid lines depict X_{t+1} as a function of X_t. Thus, whenever the solid line is below the dotted, there is a depletion of reserves and vice versa. The range for X is consistent with that we find in our simulations. Within this range, the country is always accumulating reserves prior to a sudden stop (panel a). This needs not be the case, as for a very long period without sudden stop the country could eventually reach a stationary point of reserves. Typically, however, countries run into sudden stops much before reaching such point. Panel (b) shows that during sudden stops the country always depletes reserves. It also shows that it sometimes runs out of reserves, as the solid line intersects the horizontal axis to the right of zero.

Returning to panel (a), note that the difference between the solid line and the 45 degree line is small and it shrinks as reserves rise. This means that the country accumulates reserves slowly and it reinforces the point that the country is likely to have reserves levels substantially lower than the stationary point. The

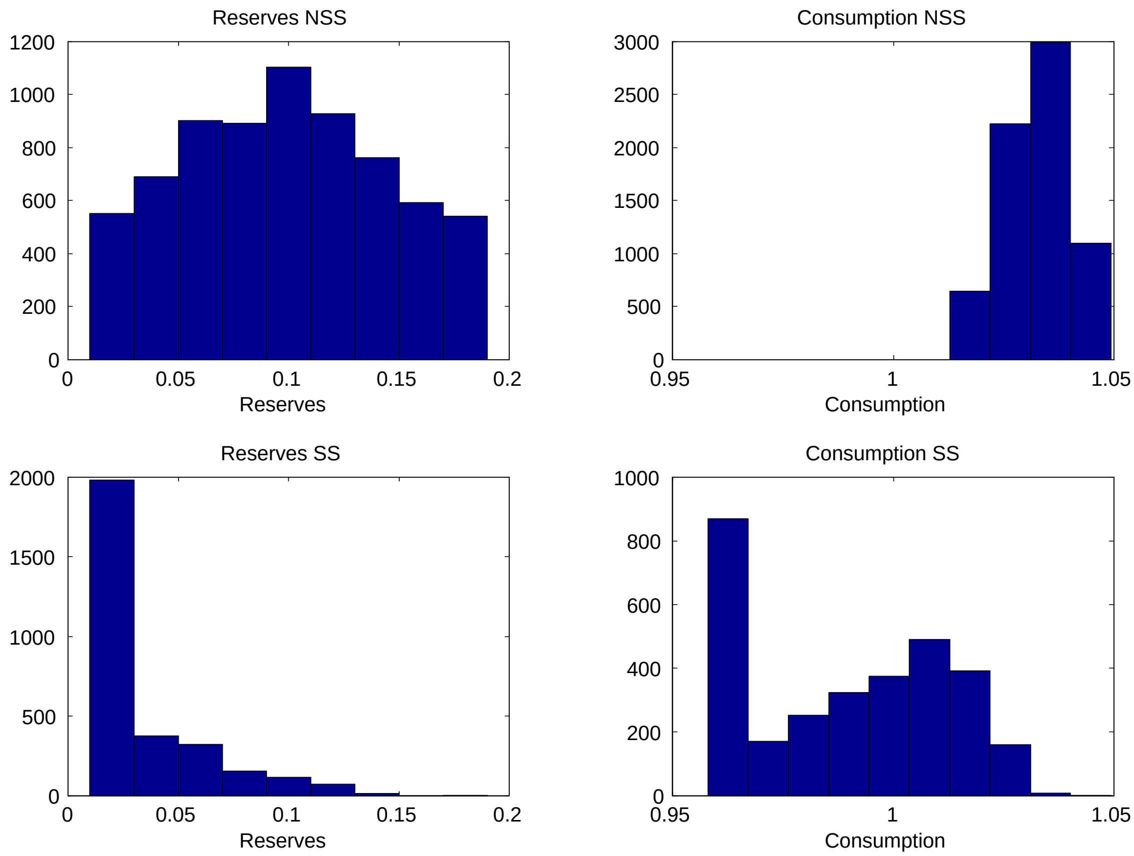

Figure 3: Histograms of reserves and consumption during SS and NSS periods. The averages were computed using 10,000 Monte Carlo simulations, each over a 15-period time interval.

next figure confirms this claim.

Figure 3 shows histograms of simulations produced by the model. Prior to sudden stops, the average level of reserves is around ten perecent and the corresponding consumption level about 1.04. During the sudden stop, on the other hand, we see a substantially lower average level of consumption. Moreover, there is spike in the distribution of consumption at around 0.96, which corresponds to the spike at zero in the histogram of reserves during sudden stops. This spike accumulates all the cases where sudden stops end up being longer than average and the country depletes the reserves in the interim. It is apparent from this figure that realistic levels of reserves accumulation do not provide a good sudden stop mechanism, especially for long lasting ones.

The next two figures further elaborate on these points. They plot consumption and reserves for three different scenarios. In the first scenario, both the pre-sudden stop phase (NSS) and the sudden stop itself last exactly as expected. Note that in this situation just enough reserves are accumulated during the NSS phase that their depletion during the sudden stop phase leads them very close to zero. The scenario depicted in the middle of the panel provides the explanation for the left tail in Figure 3. When the sudden stop is longer than expected, the country typically depletes reserves entirely before the sudden stop ends. At this point consumption is equal to y^{SS} (plus base GDP) and X_t stay at zero until the sudden stop ends. The opposite is true if the NSS phase is long and the sudden stop is short lived. In such case, substantial consumption smoothing can be achieved.

Let us now turn to the effect of time variation in the likelihood of a sudden stop. For this, we approximates the diffusion process in the model for a stationary discrete time Markov chain with only three possible values for the logarithm of λ_t. In order to obtain these states we first compute the long run mean for the log of the predicted hazard rate and its stationary standard deviation. Then we chose points corresponding to two (stationary) standard deviations below and above the mean, to obtain values of λ of 0.04, 0.17 and 0.37. We determine the transitions between these states, matching the implied conditional means and standard deviations implied by the estimated AR(1) process.[15]

Figure 6 shows the policy functions $C(X)$ and $C^{SS}(X)$, conditional on the state of the hazard. The main point of this section is the drop in the consumption function as the the sudden stop risk rises. Prior to a sudden stop, such a rise leads to what have termed a "precautionary recession." That is, a drop in consumption in *anticipation* of a sudden stop. During the sudden stop, such rise also reduces consumption even when $\tilde{\lambda}$ remains constant, because it makes the next recovery likely to be more short lived, and hence raises the value of slowing down the pace at which reserves are used.[16] The corresponding dynamics of reserves in these different regimes are illustrated in figures 7. Here a deterioration in the signal is reflected in an upward shift in the function mapping X_t onto X_{t+1}. In fact, when the likelihood of a sudden stop is very low, the country uses some of its reserves prior to any sudden stop (panel a).

Finally, figure 8portrays the sample path of reserves and consumption as an economy goes from a scenario where sudden stops are unlikely to one in which the likelihood is high, and eventually falls into a sudden stop. In particular, it spends, in the following order, three periods at λ_{min}, one at $\bar{\lambda}$, one at λ_{max}, and then three periods in the sudden stop. The main point of the figure is that the precautionary recession phase can produce significant declines in consumption, even larger than the initial impact of the actual sudden stop.

6.3 Hedging

MISSING.

[15] The exact procedure is described in Kushner and Dupuis (2000) Ch.5

[16] Of course, as reserves are depleted during the sudden stop consumption drops to y^{SS} (plus base GDP) in all states. This is the reasons the three curves "meet" close to 0.

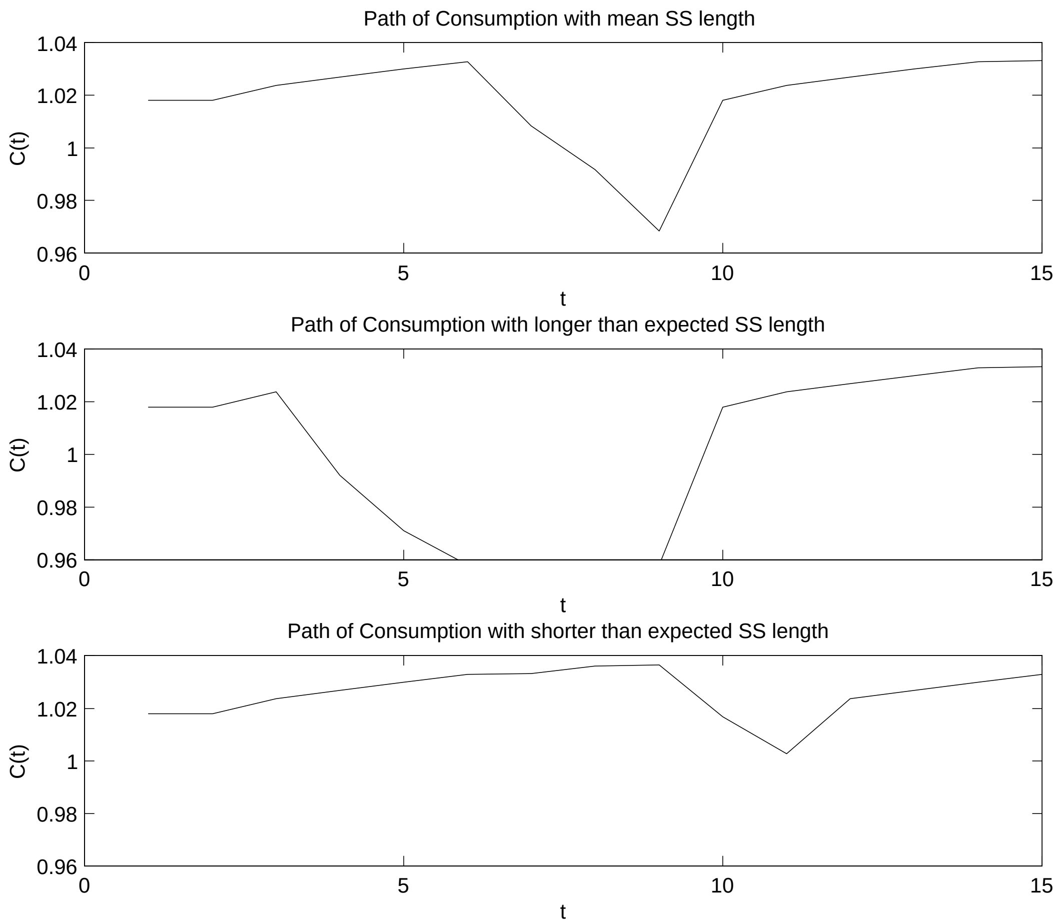

Figure 4: Path of consumption for three different scenarios. In the top panel, there is a NSS period from t=1 to t=6, followed by a SS period from t=6 to t=9 and a NSS period from t=6 to t=15. In the middle panel, the initial NSS occurs from t=1 to t=3, followed by a SS period from t=3 to t=9 and a NSS period between t=9 and t=15. In the bottom panel, the NSS period occurs from t=1 to t=9, the SS occurs from t=9 to t=11 and then there is a NSS period from t=11 to t=15.

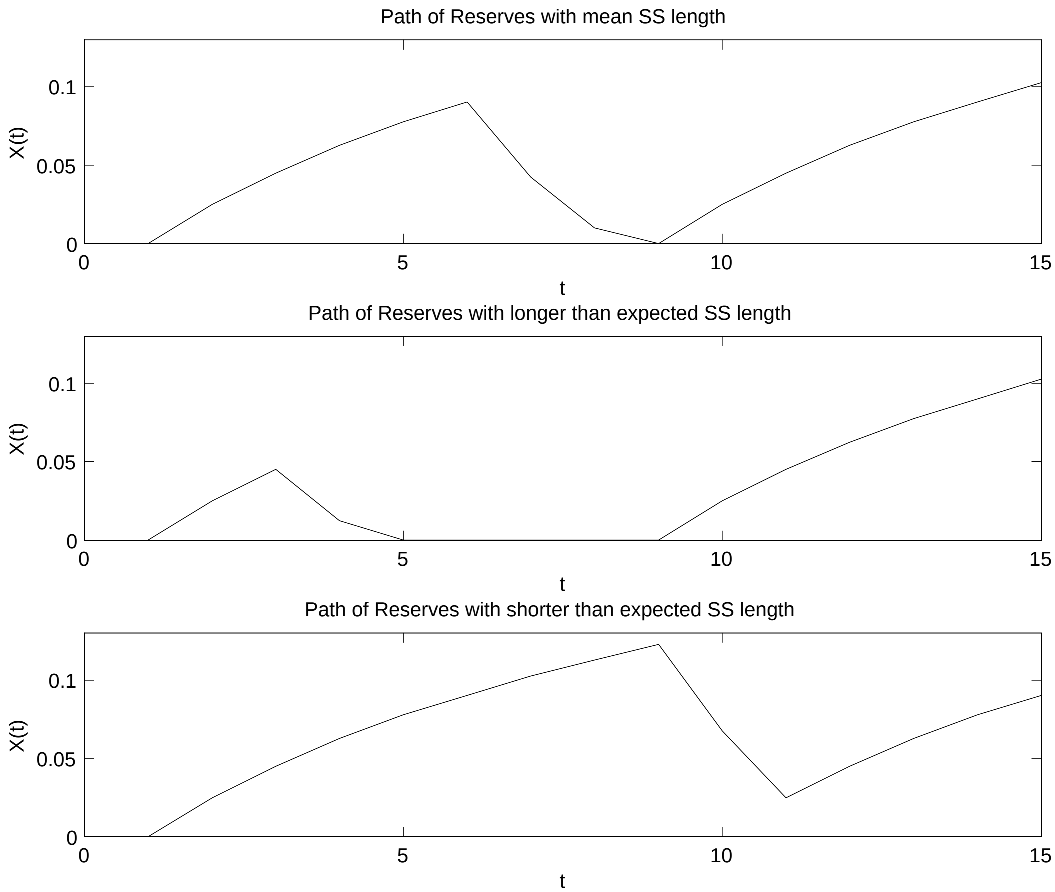

Figure 5: Path of reserves for three different scenarios. In the top panel, there is a NSS period from t=1 to t=6, followed by a SS period from t=6 to t=9 and a NSS period from t=6 to t=15. In the middle panel, the initial NSS occurs from t=1 to t=3, followed by a SS period from t=3 to t=9 and a NSS period between t=9 and t=15. In the bottom panel, the NSS period occurs from t=1 to t=9, the SS occurs from t=9 to t=11 and then there is a NSS period from t=11 to t=15.

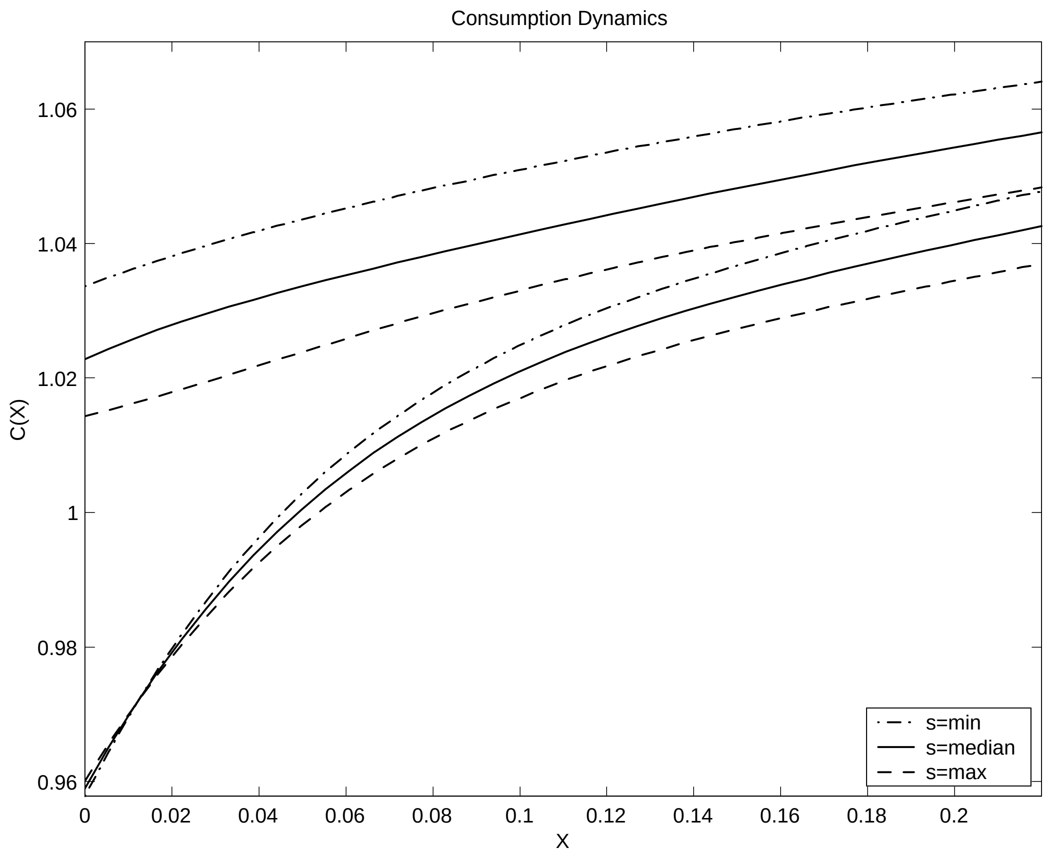

Figure 6: Consumption functions for NSS and SS and different λ_t.

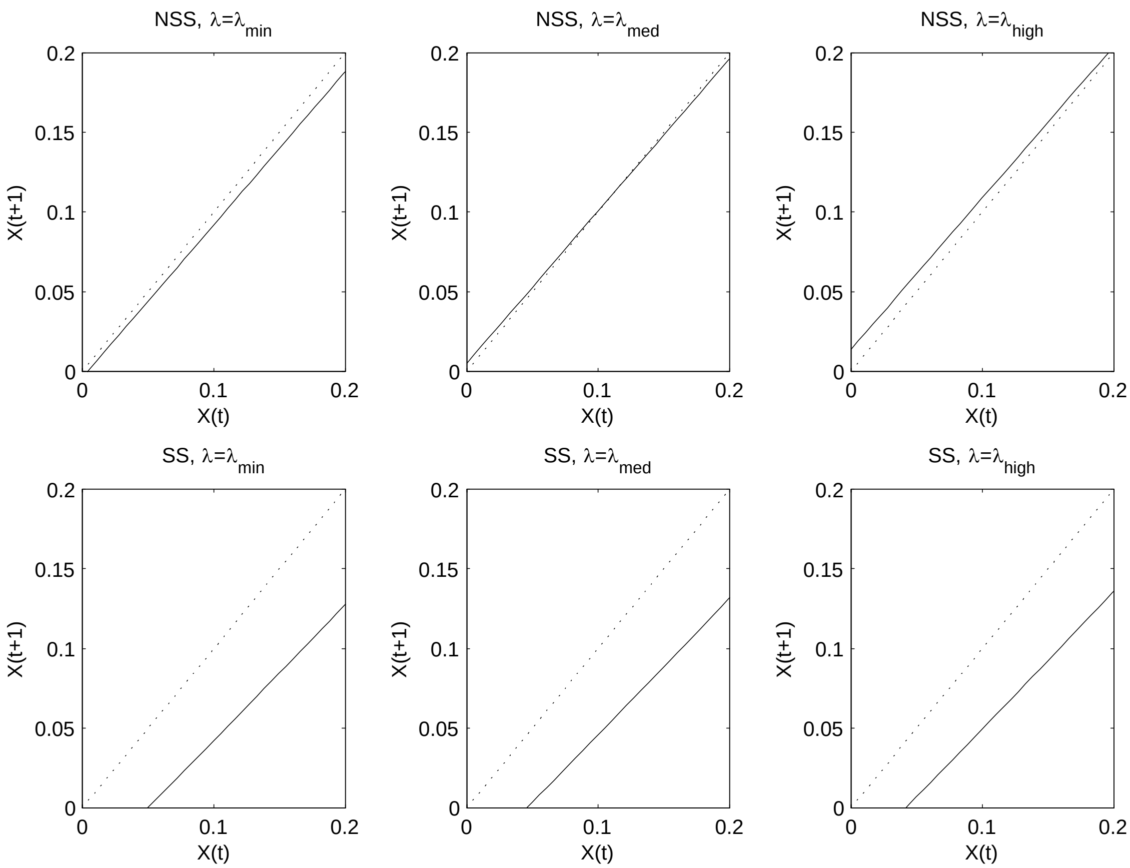

Figure 7: Transition Dynamics of Reserves.

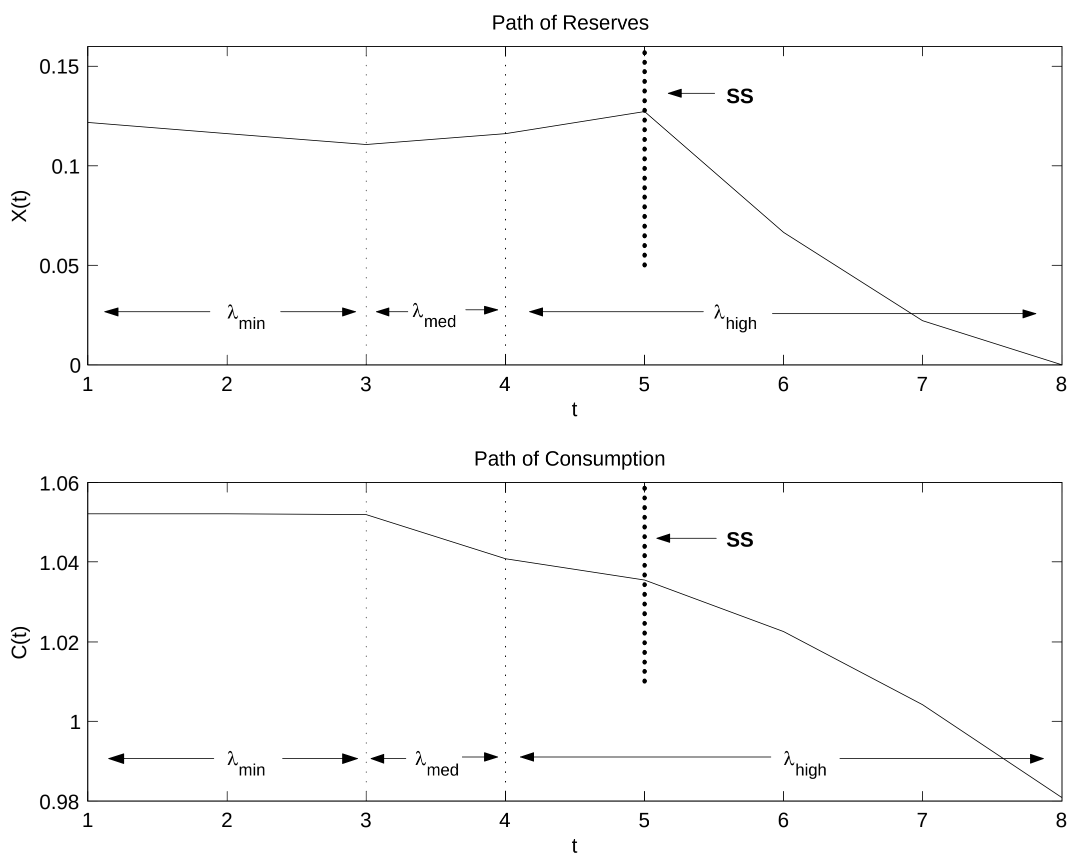

Figure 8: Consumption and reserves when hazard rates vary over time.

7 Final Remarks

Emerging market economies hold levels of international reserves that greatly exceed the levels held by developed economies (relative to their size). This would seem paradoxical given that, unlike the latter, the former face significant financial constraints with much of their growth ahead of them.

The paradox disappears once these greater financial constraints also become an important source of volatility, which countries seek to smooth. This is the context we have modelled, analyzed, and began to assess quantitatively.

Once such perspective is adopted, one must ask whether current practices, consisting primarily in accumulating non-contingent reserves, are the best countries can or should aim to do. Recasting the questions we posed in the introduction: Are there potentially less costly prudential mechanisms to deal with capital flow volatility? Who would be the countries' counterpart in these mechanisms? What is the specific role of reserves accumulation in dealing with capital flows? What kind of instruments should these reserves be invested in? How are these mechanisms and instruments limited by financial and collateral constraints?

Our framework provides aspects of an answer to each of these questions: Yes, there are potential insurance contracts, credit lines and hedging markets, that could significantly reduce the cost and improve the efficiency of mechanisms to smooth sudden stops. The natural counterpart of these instruments and contracts are not the regular emerging market specialist investors but the world capital markets at large. This is an important consideration to have in mind when designing such instruments. The much touted GDP-indexed bonds, for example, while a natural and useful instrument to trade with specialists (in fact, our swaps with specialists are a form of such bond) are unlikely to appeal to the broad markets. Non-contingent reserves have a place as well, since in practice hedges are unlikely to be perfect, and overcommitting to an imperfect hedge comes with its own risks. It is clear, nonetheless, that there are enough verifiable and contractible global variables that are significantly correlated with sudden stops and should form the basis for a better contingent strategy. Finally, the very same financial constraints that are behind these countries troubles, limit the type and amount of insurance and hedging strategies these countries can engage in. In particular, since sudden stops are mostly times when specialists are constrained as well, the strategies must be such that require little credibility and commitment on the country side. This means, essentially, policies and investments that are paid up-front rather than simple swaps of future contingencies.

Aside from the many stylized assumptions of the model, there are three substantive omissions that seem important to consider in future work. The first one is the absence of plain vanilla bonds and debt contracts. The second one is the lack of an aggregate demand mechanism that amplifies the decline in consumption following a sharp reduction in the current account deficit. The third one is the representative agent nature of the model, with a public and private sector working for each other.

We view the first omission as less limiting than it may appear as first sight. Debt, one way or the other, has always an element of contingency in it, especially when dealing with sovereigns. Thus, our swaps may not be that different from regular debt at the times that concern us. The flow of capital financing the current account, on the other hand, is new and mostly ex-post financing, and hence much harder to insure implicitly. It is perhaps for this reason that debt variables have very little relation with costly sudden stops while large current account deficits are feared by practitioners regardless of the country's indebtedness (for evidence on these see, e.g. Calvo et al (2004)). Our model is designed to isolate the current account problem. Of course, this does not mean that debt composition and levels are irrelevant considerations, but only that a significant share of the essence of the precautionary problem faced by emerging market economies can be captured in a model that removes the idiosyncrasies of debt.

The second omission is probably quantitatively important but not too hard to incorporate into the analysis in a straightforward manner. A back of the envelope calculation based on our thirteen emerging market economies suggests that the amplification factor during sudden stops may be quite large.

The third omission seems conceptually more interesting and important. Much of the central banks and governments actions in practice have to do with inducing their private sector to adopt precautionary measures that they are not naturally inclined to follow. Optimal central bank reserve management strategies need to consider the positive and negative reactions that the anticipation of such policy induce in their private sector (see, e.g. Caballero and Krishnamurthy (2003)).[17] We intend to explore decentralization issues in future work.

[17] And, of course, problems may also run counter, with private sector actions facilitating imprudent behavior by the government.

8 Appendix

We start with the proof of a fact that is generically true of the Value functions that we consider. We give the proof for the benchmark model of section 2.1, but one can apply similar arguments for all other Value functions.

Lemma 1 *The value function $V(X_t)$ is concave*

Proof. of Lemma 1 The proof follows standard arguments. First we show that the space of allowed controls is convex. To do that, consider the optimal policies associated with $X_0 = X^1$ and $X_0 = X^2$. Denote them $(c_t^1, \overline{X}_t^1)$ and the associated resulting process X_t^1. Similarly for $(c_t^2, \overline{X}_t^2)$ and X_t^2. Now consider the policy formed as the convex combination of the previous policies, i.e. take $0 \leq \alpha \leq 1$ and form $(\alpha c_t^1 + (1-\alpha)c_t^2, \alpha_t^1 \overline{X}_t^1 + (1-\alpha)\overline{X}_t^2)$. We want to show that this is an admissible policy, i.e. the resulting X_t satisfies:

$$\begin{aligned} X_t &\geq 0 \\ \lim e^{-rt} X_t &= 0 \\ \alpha_t^1 \overline{X}_t^1 + (1-\alpha)\overline{X}_t^2 &= \overline{X}_t \leq X_t \end{aligned}$$

It is immediate to see that all the above equations are true because it is easy to show that: $X_t = \alpha X_t^1 + (1-\alpha) X_t^2$ and X_t^1, X_t^2 individually satisfy the above equations (since they correspond to admissible policies). Now define:

$$J(c_t, X_t) = E\left(\int_t^{\tau^G} e^{-r(s-t)} u(c_s) ds + e^{-r(\tau^G - t)} V^G(X_{\tau^G} - \overline{X}_{\tau^G}) \right)$$

$$\begin{aligned} V(X_t) &= J(c_t^*, \overline{X}_t^*) \geq J(c_t, \overline{X}_t) = \\ &= E\left[\int_t^{\tau^G} e^{-r(s-t)} u(ac_s^1 + (1-a)\, c_s^2) ds + e^{-r(\tau^G - t)} V^G(a(X_{\tau^G}^1 - \overline{X}_{\tau^G}^1) + (1-\alpha)(X_{\tau^G}^2 - \overline{X}_{\tau^G}^2)) \right] \geq \\ &\geq aE\left[\int_t^{\tau^G} e^{-r(s-t)} u(c_s^1) ds + e^{-r(\tau^G - t)} V^G((X_{\tau^G}^1 - \overline{X}_{\tau^G}^1)) \right] + \\ &\quad + (1-a)\, E\left[\int_t^{\tau^G} e^{-r(s-t)} u(c_s^2) ds + e^{-r(\tau^G - t)} V^G((X_{\tau^G}^2 - \overline{X}_{\tau^G}^2)) \right] \\ &= \alpha V(X_t^1) + (1-\alpha) V(X_t^2) \end{aligned}$$

■

Proof. of proposition 1 The Bellman equation for the problem is given by

$$0=\max_{c_t,0\leq\overline{X}\leq X_t}\left\{u(c_t)-rV+g(V^G(X_t-\overline{X})-V)+V_X\left(rX_t-c_t+y+g\overline{X}\right)\right\}$$

or by adjoining a Lagrange multiplier ϕ

$$0=\max_{c_t,0\leq\overline{X}}\left\{u(c_t)-rV+g(V^G(X_t-\overline{X})-V)+V_X\left(rX_t-c_t+y+g\overline{X}\right)-\phi(\overline{X}-X)\right\}$$

The optimality conditions are

$$u'(c_t) = V_X \tag{19}$$

$$gV_X^G(X_t-\overline{X}) = gV_X-\phi \tag{20}$$

Write the Bellman equation as

$$0=\max_{c_t}\left\{u(c_t)-c_tV_X\right\}+\max_{\overline{X}}\left\{gV^G(X_t-\overline{X})+g\overline{X}V_X-\phi\overline{X}\right\}-(r+g)\,V+V_X\,(rX_t+y)+\phi X$$

Differentiate w.r.t. X_t using the envelope theorem

$$\begin{aligned}0 &= -c_tV_{XX}+gV_X^G(X_t-\overline{X})+gV_{XX}-(r+g)\,V_X+V_{XX}\,(rX_t+y)+V_Xr+\phi=\\ &= V_{XX}\left(rX_t-c_t+y+g\overline{X}\right)+gV_X^G(X_t-\overline{X})-(r+g)\,V_X+V_Xr+\phi\end{aligned}$$

Using the fact that

$$\frac{d\,(V_X)}{dt}=V_{XX}\frac{dX}{dt}=V_{XX}\left(rX_t-c_t+y+g\overline{X}\right)$$

Letting $\mu_t=V_X$ we get by equation (20)

$$\begin{aligned}\frac{d\,(V_X)}{dt} &= \frac{d\mu_t}{dt}=(r+g)\,\mu-\left(gV_X^G(X_t-\overline{X})+\mu r+\phi\right)=\\ &= 0\end{aligned}$$

By (19) we get that $X_t=const.$, hence

$$dX_t=\left[rX_t+g\overline{X}_t-c_t+y\right]=0$$

If $\phi>0$, then $\overline{X}_t=X_t$ and the conclusion of the proposition follows. [18] ■

Proof. of Proposition 2. To prove this proposition, notice that once the sudden stop occurs, the value function is (we assume no reserves for now):

$$V^{SS}(h)=\frac{u(y^{SS}+h)+\widetilde{\lambda}V(p)+gV^G((1-z)Y)}{r+\widetilde{\lambda}+g} \tag{21}$$

[18] In the proof we implicitly assumed twice differentiability of the Value function, to make the exposition simple. This condition can easily be relaxed by using a straightforward extension of the maxiumu principle to arrive directly at (**??**). See e.g. Leonard and Van Long (1992) Ch.10 or Yong and Zhou (1999), Ch. 5

whereas prior to the sudden stop the value function is:

$$V(p) = \frac{u(y-p) + \lambda V^{SS}(h) + gV^{G}((1-z)Y)}{r+\lambda+g} \tag{22}$$

Solving for $V(p)$ from (21) and (22), using (1) to express h in terms of p, and finally maximizing with respect to p, yields:

$$\begin{aligned} u'(y-p) &= u'(y^{SS}+h) \\ c_{\tau^{SS-}} &= y-p = y^{SS}+h = c_{\tau^{SS+}} \end{aligned}$$

where $c_{\tau^{SS-}}$ is shorthand for $\lim_{t\to\tau^{SS-}} c_t$ and similarly for $c_{\tau^{SS+}}$. From this expression we obtain the payment h that smooths consumption:

$$h = \frac{r+\widetilde{\lambda}+g}{r+\widetilde{\lambda}+\lambda+g}(y-y^{SS}).$$

■

Proof. of proposition 3 Similar to proposition 1 the country's Bellman Equation prior to the SS is

$$\begin{aligned} 0 &= \max_{c,\xi,\overline{X}} \left\{ u(c_t) + \lambda V^{SS}(X+\xi) + g\left(V^{G}(X-\overline{X}) - V\right) + \mathcal{A}V - (r+\lambda)V - \phi(\overline{X}-X) \right\} \\ \mathcal{A}V &= V_X\left(rX + y - \lambda\xi + g\overline{X} - c\right) \end{aligned}$$

where $V^{SS}(X+\xi)$ is the Value function once we enter the SS which in turn satisfies the Bellman equation:

$$\begin{aligned} 0 &= \max_{c,\xi,\overline{X}} \left\{ u(c_t) + \widetilde{\lambda}V(X) + g\left(V^{G}(X-\overline{X}) - V^{SS}\right) + \mathcal{A}V^{SS} - (r+\widetilde{\lambda})V^{SS} - \phi^{SS}(\overline{X}-X) \right\} \\ \mathcal{A}V &= V_X^{SS}\left(rX + y^{SS} + g\overline{X} - c\right) \end{aligned}$$

Computing the first order conditions yields

$$u'(c_t) = V_X \tag{23}$$

$$\lambda V_X^{SS}(X+\xi) = \lambda V_X(X) \tag{24}$$

$$gV_X^{G}(X-\overline{X}) = gV_X - \phi \tag{25}$$

and

$$\begin{aligned} u'(c_t) &= V_X^{SS} \qquad (26) \\ gV_X^{G}(X-\overline{X}) &= gV_X^{SS} - \phi^{SS} \end{aligned}$$

Equation (24) demonstrates that the country will enter enough contracts ξ so that the the marginal utilities of consumption will be equalized pre- and post- sudden stops. Thus, by equations (24), (23) and (26) we conclude that there will be no jumps in consumption once the SS occurs. Moreover there is never going to be positive reserve accumulation (i.e. $dX_t = 0$). To see this, it is easiest to think of either the optimal control formulation of the problem or proceed as in proposition 1 to arrive at the joint dynamics of $\mu_t = V_X$ and X_t

$$\begin{aligned} \frac{d\mu_t}{dt} &= (r+g)\,\mu - \left(gV_X^G(X_t - \overline{X}) + \mu r + \phi\right) - \lambda(V_X^{SS}(X+\xi) - \mu_t) \qquad (27) \\ \frac{dX_t}{dt} &= rX_t - c_t + y + g\overline{X} - \lambda\xi \end{aligned}$$

Plugging (25) and (24) into (27) gives

$$\frac{d\mu_t}{dt} = 0$$

and thus we can replicate the arguments of proposition 1 to show that $dc_t = 0, X_t = const.$ prior to the SS. It is important to stress that this result is *not true* during the SS (unless there are SS-anuities of the type defined in the text). We discuss the behavior of consumption and reserves during the SS in proposition 4 ■

.

Proof. of proposition 4 Define X^* as:

$$X^* = \frac{Y(1-z) - y^{SS}}{r+g} \qquad (28)$$

We will consider two regions. First we are going to assume that $0 \leq X_0 < X^*$ and we will show that there is reserve accumulation ($dX_t > 0$) prior to the SS and reserve decumulation ($dX_t < 0$) during the SS. However, if the level of reserves is at X^* or above, then there will be no more accumulation of reserves, and c_t will remain constant at the level $Y(1-z)$:

$$c_t = Y(1-z), \text{ if } X_0 \geq X^* \qquad (29)$$

The Bellman equation prior to the SS is given by

$$0 = \max_{c_t, \overline{X}} \left\{u(c_t) - rV + g(V^G(X_t - \overline{X}) - V) + \lambda\left(V^{SS} - V\right) + V_X\left(rX_t - c_t + y + g\overline{X}\right) - \phi_1(\overline{X} - X)\right\} \qquad (30)$$

whereas during the SS the Bellman equation becomes

$$\begin{aligned} 0 &= \max_{c_t, 0 \leq \overline{X} \leq X_t} \left\{u(c_t) + gV^G(X_t - \overline{X}) + \widetilde{\lambda}V + V_X^{SS}\left(rX_t - c_t + y^{SS} + g\overline{X}\right) - \phi_2(\overline{X} - X)\right\} \qquad (31) \\ &\quad -(r + g + \widetilde{\lambda})V^{SS} \qquad (32) \end{aligned}$$

with optimality conditions

$$\begin{aligned} u'(c_t) &= V_X \qquad (33) \\ gV_X^G(X_t - \overline{X}) &= gV_X - \phi_1 \qquad (34) \end{aligned}$$

and

$$u'(c_t) = V_X^{SS} \quad (35)$$
$$gV_X^G(X_t - \overline{X}) = gV_X^{SS} - \phi_2 \quad (36)$$

Differentiating the Bellman equation prior to the SS, and using the Envelope Theorem we get

$$\begin{aligned} 0 &= -c_t V_{XX} - (r+g+\lambda)V_X + gV_X^G\left(X - \overline{X}\right) + \lambda V_X^{SS} + V_{XX}\left(rX_t - c_t + y + g\overline{X}\right) + V_X r + \phi_1 \\ 0 &= -\lambda V_X + \lambda V_X^{SS} + V_{XX}\left(rX_t - c_t + y + g\overline{X}\right) = -\lambda V_X + \lambda V_X^{SS} + \frac{d\left(V_X\right)}{dt} \end{aligned}$$

Thus -prior to the SS- we obtain

$$\frac{\frac{dc_t}{dt}}{c_t} = -\frac{1}{\gamma}\lambda\left(1 - \frac{V_X^{SS}}{V_X}\right) \quad (37)$$

By identical derivations we get for the dynamics of consumption and wealth during the SS:

$$\frac{\frac{dc_t}{dt}}{c_t} = -\frac{1}{\gamma}\widetilde{\lambda}\left(1 - \frac{V_X}{V_X^{SS}}\right) \quad (38)$$

Thus if we can show that $V_X < V_X^{SS}$, we will obtain $dc_t > 0$ prior to the SS and $dc_t < 0$ during the SS. Moreover by the concavity of V and the conditions (33) and (35) we will obtain $dX > 0$ prior to the SS and $dX < 0$ thereafter. Thus we just need to show that $V_X < V_X^{SS}$. To see that, assume otherwise. Also assume first that both $\phi_1, \phi_2 > 0$. Then $V^G\left(X_t - \overline{X}\right) = V^G(0)$, whether we are in a SS or not. Compute now the difference between the left sides of (30) and (31). Observe moreover that according to our (counterfactual) assumption $c^{SS} > c$ by (33) and (35). By the monotonicity of $u()$ this implies $u(c^{SS}) > u(c)$. Moreover, as long as we allow for free disposal it has to be the case that $V > V^{SS}$ (since the difference between the stage prior to the SS and during is the presence of an extra $y - y^{SS}$, the country could always "destroy" that difference) Combining these arguments we get:

$$0 = \left[u(c) - u(c^{SS})\right] - \left(r + g + \lambda + \widetilde{\lambda}\right)\left(V - V^{SS}\right) + V_X\left((r+g)X_t - c_t + y\right) - V_X\left((r+g)X_t - c_t + y^{SS}\right)$$

However, notice that although the left hand side of this equation is 0, the right hand side is clearly negative according to our assumptions, since $u(c^{SS}) > u(c)$, $V - V^{SS} > 0$, and according to (37) and (38) $dX = (r+g)X_t - c_t + y < 0$ prior to the sudden stop and $(r+g)X_t - c_t + y^{SS} > 0$ during the SS. This establishes the contradiction if $\phi_1, \phi_2 > 0$. Assuming $\phi_2 = 0$, $\phi_1 > 0$ just adds a term of the sort $V^G(0) - V^G(x)$ with $x > 0$ which is clearly negative and if both ϕ_1 and $\phi_2 = 0$, we still obtain a negative term because (34) and (36) imply that during the SS the country will pledge a lower $\overline{X}$ under our counterfactual and thus $V^G\left(X - \overline{X}^{SS}\right) > V^G\left(X - \overline{X}\right)$, where $\overline{X}^{SS}$ denotes the wealth pledged during the SS and $\overline{X}$ prior to it. This establishes the contradiction for all values of ϕ, and the assertion of the proposition, for $X < X^*$

We know turn to the determination of a steady state of reserves, beyond which accumulation doesn't make sense. At a steady state of the system we must have that:

$$V_X^{SS} = V_X \tag{39}$$

and thus:

$$\begin{aligned} u'(c_{\tau^{SS-}}) &= u'(c_{\tau^{SS+}}) \\ c_{\tau^{SS-}} &= c_{\tau^{SS+}} \end{aligned} \tag{40}$$

Moreover by evaluating equations (34) and (36) at the steady state we get that:

$$\phi_1 = \phi_2 = 0$$

since otherwise it would have to be the case that $c_{\tau^{SS-}} = (r+g)X_{\tau^{SS}} + y$ and $c_{\tau^{SS+}} = (r+g)X_{\tau^{SS}} + y^{SS}$, and so (40) cannot hold. Accordingly, we know by (39), (34) and (36) that at the steady state, the constraint $\overline{X} \leq X_t$ will be "just binding" inside the sudden stop and not binding prior to the sudden stop. In other words

$$c_{\tau^{SS+}} = (r+g)X_{\tau^{SS}} + y^{SS} = (r+g)X^* + y^{SS} \tag{41}$$

where the second equality follows because we are at a steady state and thus for any stopping time $X_\tau = X^*$.Moreover by the optimality condition (36) it will be the case that:

$$u'(c_{\tau^{SS+}}) = V_X^G(0)$$

and thus

$$c_{\tau^{SS+}} = Y(1-z) \tag{42}$$

Thus by (41) and (42) we get (28), and since $c_{\tau^{SS+}} = c_t$ for all t in a steady state we get equation (29). It is trivial to show that for $X > X^*$ the equations (33), (35), (34), and (36) imply that the country has enough collateral to completely avoid sudden stops.

The assertion that reserves can be depleted in finite time comes from evaluating the $\frac{V_X}{V_X^{SS}}$ for $X_t = 0$, and observing that[19] $V_X^{SS} = \left(y^{SS}\right)^{-\gamma}$, else we would have $dX < 0$ and the constraint $X_t \geq 0$ would be violated. Contrary to that V_X will be strictly lower than $\left(y^{SS}\right)^{-\gamma}$, as can be proved by an argument identical to the one we gave above when we were comparing the two derivatives of V and V^{SS}. Thus $\lim_{X_t \to 0} \frac{V_X}{V_X^{SS}} < 1$ and the constraint $X_t = 0$ will typically be hit in finite time. ■

Proof. of proposition 5 We provide a sketch only, since the argument for the first assertion has been

[19] The continuity of the controls around $X_t = 0$ can be shown by an argument identical to Fleming and Soner (1993), section II.12

developed in the text. Thus we focus only on the latter part. By arguments that are virtually identical to the ones in the proof of proposition 4, we obtain that:

$$\begin{aligned}\frac{dc_t}{c_t} &= -K_1(X_t)dt \\ dX_t &= \left(rX_t - c_t + y^{SS} - \xi_t p\right)dt\end{aligned}$$

with

$$K_1(X_t) = \frac{\lambda}{\gamma}\left(1 - \frac{EV_X^{SS}(X_t + \xi_t\zeta)}{V_X(X_t)}\right) < 0$$

by equation (10) we that consumption will be increasing, and by the condition:

$$V_X = u'(c)$$

we get that $dX_t > 0$. The proof of depletion of reserves during the sudden stop is identical to the proof in proposition 4. ■

Proof. of proposition 6 We focus on the pre-sudden stop Bellman equation. The arguments for the Bellman equation during the SS follow identical steps. Note that the Bellman equation is given as

$$\begin{aligned}0 &= \max_{c,\overline{X}\leq X_t}\left\{u(c_t) + \lambda(s_t)V^{SS}(X,s) + g\left(V^G(X_t - \overline{X}) - V(X,s)\right) - (r + \lambda(s_t))V(X,s) + A^R V\right\} \\ A^R V &= V_X(X,s)\left(rX_t + g\overline{X} + y - c\right) + \mu(s_t)V_s(X,s) + \frac{1}{2}\sigma(s_t)^2 V_{ss}(X,s) - \phi(\overline{X} - X)\end{aligned}$$

The FOC's are

$$u'(c_t) = V_X \tag{43}$$

$$gV_X^G(X_t - \overline{X}) = gV_X - \phi \tag{44}$$

Differentiating the Bellman equation w.r.t. X_t we get

$$\begin{aligned}0 = {} & \lambda(s_t)\left(V_X^{SS} - V_X\right) + \left[g\left(V_X^G(X - \overline{X}) - V_X\right) + \phi\right] - rV_X + rV_X \\ & + V_{XX}\left(rX_t + y - c + g\overline{X}\right) + \mu(s_t)V_{Xs} + \frac{1}{2}\sigma(s_t)^2 V_{Xss}\end{aligned} \tag{45}$$

Applying Ito's Lemma to (43) we also get

$$\begin{aligned}dV_X = {} & \left(V_{XX}\left(rX_t + y - c + g\overline{X}\right) + V_{Xs}\mu(s_t) + \frac{1}{2}\sigma(s_t)^2 V_{Xss}\right)dt + \\ & + V_{Xs}\sigma(s_t)dB_t\end{aligned} \tag{46}$$

Combining (45) with (46) and (44) we get

$$dV_X = \left[-\lambda(s_t)\left(V_X^{SS} - V_X\right)\right]dt + V_{Xs}\sigma(s_t)dB_t$$

Now, applying Ito's Lemma to

$$u'(c_t) = V_X$$

which for CRRA preferences becomes

$$c_t = V_X^{-\frac{1}{\gamma}} \tag{47}$$

we obtain

$$\begin{aligned} dc_t &= \left(-\frac{1}{\gamma} V_X^{-\frac{1}{\gamma}} \frac{-\lambda(s_t)\left(V_X^{SS} - V_X\right)}{V_X} + V_X^{-\frac{1}{\gamma}} \frac{1}{2} \left(\frac{V_{Xs}\sigma(s_t)}{V_X} \right)^2 \left(\frac{\gamma+1}{\gamma^2} \right) \right) dt \\ & -V_X^{-\frac{1}{\gamma}} \frac{\sigma(s_t)}{\gamma} \frac{V_{Xs}}{V_X} dB_s \end{aligned}$$

which implies by (47)

$$\begin{aligned} \frac{dc_t}{c_t} &= \left(-\frac{1}{\gamma} \lambda(s_t) \left(1 - \frac{V_X^{SS}}{V_X} \right) + \frac{1}{2} \left(\frac{V_{Xs}\sigma(s_t)}{V_X} \right)^2 \left(\frac{\gamma+1}{\gamma^2} \right) \right) dt \\ & -\frac{\sigma(s_t)}{\gamma} \frac{V_{Xs}}{V_X} dB_s \end{aligned}$$

■

Proof. of proposition 7 We focus on the pre-sudden stop Bellman equation. The arguments for the Bellman equation during the SS follow identical steps. Note that the Bellman equation is given as

$$\begin{aligned} 0 &= \max_{c, \overline{X} \leq X_t} \left\{ u(c_t) + \lambda(s_t) V^{SS}(X,s) + g\left(V^G(X_t - \overline{X}) - V(X,s) \right) - (r + \lambda(s_t)) V(X,s) + AV \right\} \\ AV &= V_X(X,s)\left(rX_t + g\overline{X} + y - c \right) + \mu(s_t) V_s(X,s) + \frac{1}{2}\sigma(s_t)^2 V_{ss}(X,s) - \phi(\overline{X} - X) \\ & + \frac{1}{2}\sigma_F^2 V_{XX} F_t^2 \pi^2 - \pi V_{Xs} \sigma_F F \sigma(s_t) \end{aligned}$$

The FOC's are

$$u'(c_t) = V_X \tag{48}$$

$$gV_X^G(X_t - \overline{X}) = gV_X - \phi \tag{49}$$

$$\pi = \frac{V_{Xs}\sigma(s_t)}{V_{XX} F_t \sigma_F} \tag{50}$$

Differentiating the Bellman equation w.r.t. X_t we get

$$\begin{aligned} 0 &= \lambda(s_t)\left(V_X^{SS}(X + \xi) - V_X \right) + \left[g\left(V_X^G(X - \overline{X}) - V_X \right) + \phi \right] - rV_X + rV_X \\ & + V_{XX}\left(rX_t + y - c + g\overline{X} - \lambda(s_t)\xi \right) + \frac{1}{2}\sigma_F^2 V_{XXX} F_t^2 \pi^2 \\ & - \pi V_{XXs} \sigma_F F \sigma(s_t) + \mu(s_t) V_{Xs} + \frac{1}{2}\sigma(s_t)^2 V_{Xss} \end{aligned} \tag{51}$$

Applying Ito's Lemma to (48) we also get

$$dV_X = Adt + CdB_t$$

where

$$C = -V_{XX}\pi\sigma_F F + V_{XS}\sigma(s_t) = 0$$

and

$$\begin{aligned} A &= V_{XX}\left(rX_t + y - c + g\overline{X} - \lambda(s_t)\xi\right) + \frac{1}{2}\sigma_F^2 V_{XXX} F_t^2 \pi^2 \\ &\quad -\pi V_{XXs}\sigma_F F\sigma(s_t) + \mu(s_t)V_{Xs} + \frac{1}{2}\sigma(s_t)^2 V_{Xss} \end{aligned} \tag{52}$$

Combining (51),(52), and using (49), (50) we get

$$dV_X = \left[-\lambda(s_t)\left(V_X^{SS} - V_X\right)\right]dt$$

Now, applying Ito's Lemma to

$$c_t = V_X^{-\frac{1}{\gamma}}$$

we obtain

$$\frac{dc_t}{c_t} = -\frac{\lambda(s_t)}{\gamma}\left(1 - \frac{V_X^{SS}}{V_X}\right)dt$$

■

Proof. of Proposition 8 The steps are very similar to the ones in proposition 7. The Bellman equation prior to the SS becomes:

$$\begin{aligned} 0 &= \max_{\xi,c,\pi,0\leq\overline{X}\leq X_t,}\left\{u(c_t) + \lambda(s_t)\left[V^{SS}(X+\xi) - V\right] + g\left(V^G(X-\overline{X}) - V\right) - rV + A^R V\right\} \\ A^R V &= V_X\left(rX_t + y - c + g\overline{X} - \lambda(s_t)\xi\right) + \frac{1}{2}\sigma_F^2 V_{XX} F_t^2 \pi^2 \\ &\quad -\pi V_{Xs}\sigma_F F\sigma(s_t) + \mu(s_t)V_s + \frac{1}{2}\sigma(s_t)^2 V_{ss} - \phi(\overline{X} - X) \end{aligned}$$

The FOC's are:

$$u'(c_t) = V_X \tag{53}$$

$$V_X^{SS}(X+\xi) = V_X \tag{54}$$

$$gV_X^G(X_t - \overline{X}) = gV_X - \phi \tag{55}$$

$$\pi = \frac{V_{Xs}\sigma(s_t)}{V_{XX}F_t\sigma_F} \tag{56}$$

First we differentiate the Bellman equation w.r.t. X using the envelope theorem. We get:

$$\begin{aligned} 0 &= \lambda(s_t)\left(V_X^{SS}(X+\xi)-V_X\right)+\left[g\left(V_X^G(X-\overline{X})-V_X\right)+\phi\right]-rV_X+rV_X \\ &+V_{XX}\left(rX_t+y-c+g\overline{X}-\lambda(s_t)\xi\right)+\frac{1}{2}\sigma_F^2 V_{XXX}F_t^2\pi^2 \\ &-\pi V_{XXs}\sigma_F F\sigma(s_t)+\mu(s_t)V_{Xs}+\frac{1}{2}\sigma(s_t)^2 V_{Xss} \end{aligned} \tag{57}$$

Also by Ito's Lemma:

$$dV_X = Adt + CdB_t$$

where

$$C = -V_{XX}\pi\sigma_F F + V_{XS}\sigma(s_t) = 0$$

by (56) and

$$\begin{aligned} A &= V_{XX}\left(rX_t+y-c+g\overline{X}-\lambda(s_t)\xi\right)+\frac{1}{2}\sigma_F^2 V_{XXX}F_t^2\pi^2 \\ &-\pi V_{XXs}\sigma_F F\sigma(s_t)+\mu(s_t)V_{Xs}+\frac{1}{2}\sigma(s_t)^2 V_{Xss} \end{aligned}$$

Notice that the first line in (57) is identically 0 due to the optimality conditions (54)-(55) whereas the second and third lines are identical to A. Hence $A = 0$. Since

$$dV_X = 0$$

on an optimal path. It follows that:

$$dc_t = 0$$

by (53). It remains to show that $X_t > 0$ for all t as long as X_0 is higher than the minimum of the upper bounds given in the statement of the proposition. Notice that (by (54)):

$$\xi = V_X^{SS-1}(u'(c)) - X$$

where $V_X^{SS-1}(\cdot)$ is the inverse function of V_X^{SS}. Since $u'(c) = const.$ it will be the case that $V_X^{SS-1}(u'(c))$ can depend at most on s_t. Thus the budget constraint becomes:

$$dX_t = \left[(r+\lambda(s_t))X_t + y - c_t + g\overline{X} - \left[\lambda(s_t)V_X^{SS-1}(u'(c))\right]\right]dt - \pi_t\sigma_F F_t dB_t$$

Moreover, we can constrain attention on the set where ϕ is non-zero (i.e. all the reserves are used as collateral) w.l.o.g. which gives:

$$X_0 = (c_0 - y)E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}ds|s_0\right)+ \tag{58}$$

$$+E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}\lambda(s_t)V_X^{SS-1}(u'(c_0))ds|s_0\right) \tag{59}$$

The second term is at most:

$$c_0 E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}\lambda(s_t)ds|s_t\right)$$

so that:

$$X_0 \leq (c_0 - y)\, z(s_0) + \underbrace{c_0 E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}\lambda(s_t)ds|s_0\right)}_{B}$$

or:

$$\begin{aligned} c_0 &\geq \frac{X_0 + yz(s_0)}{z(s_0)+B} \\ c_0 - y &\geq \frac{X_0 - yB}{z(s_0)+B} \geq 0 \end{aligned}$$

by assumption 15. From this we easily get the conclusion that for any $t \in \left(0, \tau^{SS} \wedge \tau^{G}\right)$:

$$X_t = (c_0 - y)\, z(s_t) + E\left(\int_0^\infty e^{-(r+g)t+\int_0^t \lambda(s_u)du}\lambda(s_t)V_X^{SS-1}(u'(c_0))ds|s_t\right) \geq 0$$

since both terms are positive. To see the last assertion notice that:

$$\begin{aligned} X_t &= (c_0 - y)\, z(s_t) + Z(s_t) = \\ &= \frac{[X_0 - Z(s_0)]}{z(s_0)} z(s_t) + Z(s_t) \geq \\ &= \frac{X_0 + Z(s_t)z(s_0) - Z(s_0)z(s_t)}{z(s_0)} \\ &\geq 0 \end{aligned}$$

by condition (16) ■